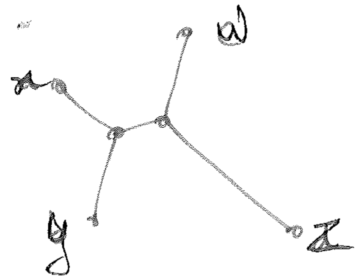
CASS LECTURES (1)



4PC

d_{xy} , xy

$d_{xy} + d_{wz} < d_{xw} + d_{yz} = d_{xz} + d_{wy}$

Existence 1. $\exists T, w$: $d = d_{(T,w)} \iff d$ satisf 4PC.

Uniqueness 2. If $w(e) > 0$ $\forall e$ int.

$d_{(T,w)} = d_{(T',w')} \Rightarrow T = T'$ & $w = w'$

History mid 1960's Russian – Zaretskii (1965)
1969 Simoes-Pereira.

Redundancy

$\Phi : \mathbb{R}^{E(T)} \to \mathbb{R}^{\binom{X}{2}}$ There exists a retraction $\Psi : \mathbb{R}^{\binom{X}{2}} \to \operatorname{Im}\phi$ but NJ is not one!

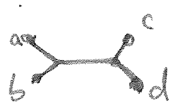
$w \longmapsto d_{(T,w)} \longrightarrow$

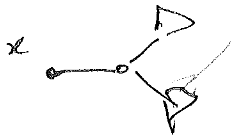
$\dim \operatorname{Im}(\phi) = |E(T)|$. $w(e) = \frac{1}{2}(\ldots)$

Eg. $n = 4$.

ab
cd
ac
bd
ad
bc

+ 3 of these (not 4)



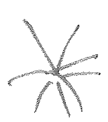


x-rooted com

M(T) matroid . max lin indep sets . min subsets that determine the edge lengths.

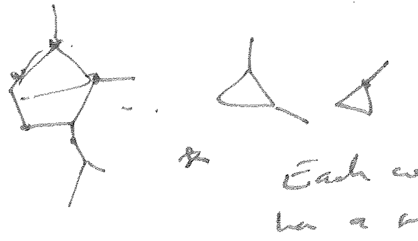
$$\begin{array}{c|c} & xy \ \ldots \\ \hline \begin{matrix} e_1 \\ e_2 \\ e_3 \\ \vdots \end{matrix} & \left[\begin{matrix} \\ \\ \end{matrix} \right] \end{array}$$

Theorem $M(T_1) = M(T_2) \Rightarrow T_1 = T_2$



star tree $M(T) =$

$(X, Z) =$



& Each component has a single odd cycle

(Omitted) ②

~~Ultrametrics~~

Spse $\forall Q \in \{ab, c, d\}$, $\exists$ tree $ab|cd$

~~Consider~~ So

$(ac+bd) - (ad+bc) = 0$ (not nec compatible)

② What is ~~dimension of~~ the space

$\Omega = \{ \underline{v} \in \mathbb{R}^{\binom{X}{2}} : \underline{v}$ satisfies all these constraints ?$\}$

$\Sigma_Q = \{A|B : \forall a,a' \in A, \; \forall b,b' \in B \;\; aa'|bb' \in Q\}$

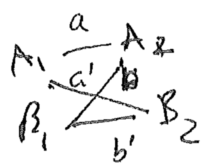
Σ_Q is pairwise compatible so forms a ~~tree~~.

$\dim \Omega = \#$ edges in this tree

$= |\Sigma_Q|$.

eg T binary. Q_T.

$A_1 | B_1 \quad A_2 | B_2$



$aa' || bb'$

$ab' | a'b$.

Over $\mathbb{R}$

$$d(x,y) = u(x,y) - v(x,y)$$

over G, $\mathbb{Z}$, $\mathbb{Z}/n$ etc. $\mathbb{R}^n$! b $\mathbb{Z}/n$ Clock

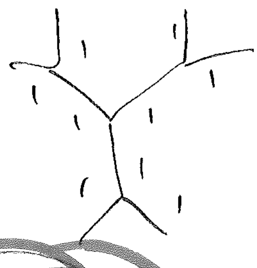
Gen 4PC

Necc cond. $d_{xy} + d_{wz} + 2\xi = d_{xw} + d_{yz} = d_{xz} + d_{wy}$

[illegible] — can't use min! (or NJ).

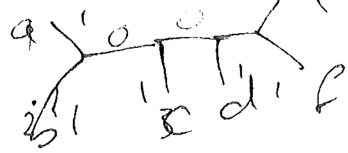
Turns out to be exactly the same condition/assumption

prov. G has no el^t of order 2



$G = \mathbb{Z}_2^k$ then d over satisfies 4PC

$\Leftrightarrow$ d has rep on every tree



G not el^t of deg 4

$\Leftrightarrow$ every d has a rep on ≥ 1 tree

Element of order 4

EXAMPLE!

Pity!

~~real~~ real data

el^t of order 2 $\to$ conj?

PD

Eg of order 4 ϵ.

$d_{ij} = 2\epsilon$, if $ij = 24\ 23\ 14$

$d_{ij} = 0$, else

d satisfies the 4PC wrt G.

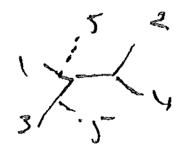
$\{1, 234\}$

$d_{13} + d_{24} + d_{14} + d_{23} = d_{12} + d_{34}$ 13|24

$\{1, 3, 4, 5\}$

35|14

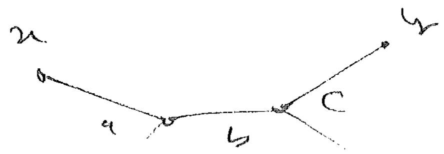
15|23.



BO

Nonabelian group ?

d_{xy}



$a+b+c = c+b+a = d_{yx}$

but what about non-abelian ?

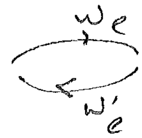
$d_{xy} = abc$

$d_{yx} = cba$ might not be equal

1st asym distance — over $\mathbb{R}$.

$d_{xy} + d_{yx}$.

$\Delta_{xy} = \frac{1}{2}(d_{xy} + d_{yx})$



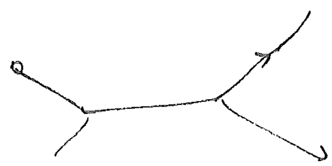
$w_e + w_e' \neq 0$.

Provided $\vec{w_e} + \overleftarrow{w_e} \neq 0$ then we're ok.

Uniqueness of $\circlearrowright$?

Relevance → transition matrices

$W_{uv} = P^{(u,v)}$



$P^{ij} = (J^{ii})^{-1} J^{ij}$

$P^{uv} P^{vu} \neq I$.

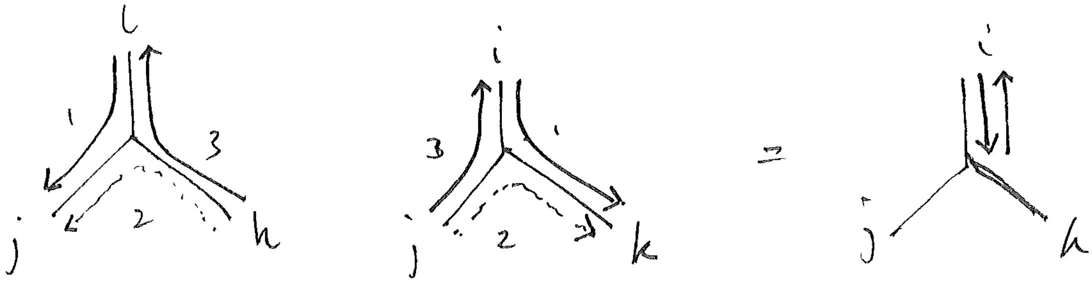
$0 < \det(P^e) < 1$

log dets:

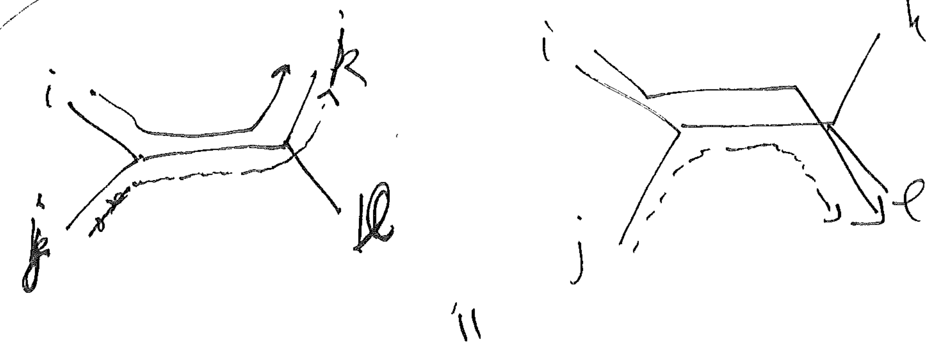
Group hom from $f: M_n \to \mathbb{R}$

$\delta_{ij}' = \frac{1}{2}(f(\delta_{ij}) + f(\delta_{ji})) = \frac{\log \det}{\sqrt{\cdots}}$

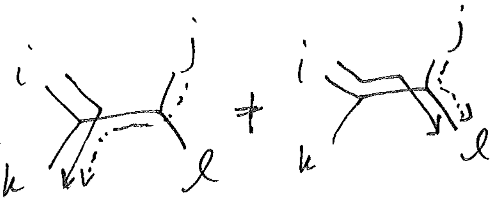
gochs lemma $(ab)^{-1} = b^{-1}a^{-1}$ B1

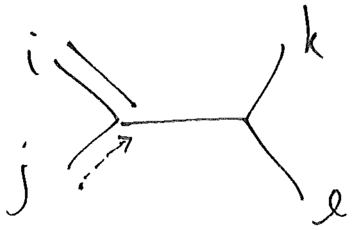


$(ij)(kj)^{-1}(ki)$
$= (ik)(jk)^{-1}(ji)$



$(ik)(jk)^{-1} = (il)(jl)^{-1}$





H_g

$H_g = \{ ik\,(jk)^{-1}(jl)(il)^{-1} \}$
$i,j,k,l \in X$

Symbolic ultrametric

return-trip metric $w(u,v)\,w(v,u) \neq 1_G$
standard

A

Evening talk

A/. M - arbitrary set

$\delta : \binom{X}{2} \longrightarrow M$. δ is a symbolic ultrametric

Note M has no order OR algebraic structure!

(U1) $|\{\delta_{ab}, \delta_{bc}, \delta_{ca}\}| \leq 2$

(U2) $\nexists\, a, b, c, d \in X$:

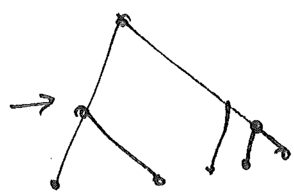
$$\delta_{ab} = \delta_{bc} = \delta_{cd} \neq \delta_{bd} = \delta_{ad} = \delta_{ac}$$

δ has a symbolic rep. over M if $\exists$ rooted phyl tree $T = (V, A)$

& map $h : V \longrightarrow M$;

$$\delta_{ab} = h(v_{ab})$$

$\uparrow$ MRCA of a & b



$h(u) \neq h(v)$ $\forall (u,v)$ int

Clearly satisfies (U1) & can check (U2).

Theorem δ is a sym. ultrametric

$\Longleftrightarrow$ δ has a sym. rep in which case tree is unique

(proved $h(u) \neq h(v)$ $\forall$ internal (u, v) edges)

& algorithm.

Basis of proof of δ-result (+ other later)

8

<u>Curious Corollary</u>

tree T on X.

Suppose each species in X has some 'feature' / key that has evolved on tree (change on every edge)

Select $z \in X$ & let

$a(x,y,z)$ = ancestral feature at median of $\mathrm{med}_T(x,y,z)$

Then from $(x,y) \longmapsto a(x,y,z)$

we can find T !

Proof $\delta_{xy} = a(x,y,z)$ is a symb. ultrametric rep. on T.

C   Elements of order 2

$d(x,y) \longmapsto d(xyz),\ d(A)$

Σ_r ~~|A|~~ $A = \{a, b, c\}$

$d(A) = \frac{1}{2}(d_{ab} + d_{bc} + d_{ac})$

in Abel group comps!

$$A \subseteq X, \quad d_{(T,w)}(A) = \sum_{e \in T(A)} w(e)$$

- 'phyl diversity' of A on T.

Over $\mathbb{R}$

Pauplin

$|A| > |B| \quad \exists x \in A - B :$

$$d(A - x) + d(B + x) \geq d(A) + d(B)$$

Strong exchange property.

Tight Span
→ Paul Tupper + David Bryant
→ Stefan + Dress.

Tree reconstruction

Pachter + Speyer

$$\phi : \binom{X}{k} \longrightarrow \mathbb{R}$$

$$A \longmapsto d_{(T,w)}(A)$$

T can be uniquely reconstructed from this map provided for each k

$$2 \leq k \leq \left(\frac{n+1}{2}\right)$$

$\therefore\ d_{(T,w)}(A) = d_{(T',w')}(A) \ \forall A \subseteq \ldots \Rightarrow T = T'$

(D)

Abelian groups : Recall El^2 of order 2 cause problems.

Solution $d(A)$ for $|A| = 2$ & $|A| = 3$.

$$\binom{X}{\leq 3} \longrightarrow G$$

$$A \longmapsto d_{(T,w)}(A)$$

determines T & w provided $w(e) \neq 1_G$ on all int edges

Proof – again by symbolic ultrametric theory.

Existence when is $d = d_{(T,\omega)}$?

d : ~~f~~

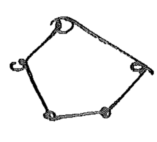
(3PC) $2d(abc) = d(ab) + d(ac) + d(bc)$

(4PC) $\left|\{\underbrace{d_x(ab)}_{d(abx)-d(ab)}, d_x(ac), d_x(bc)\}\right| \leq 2$

(4PC*).

(5PC)

Proof uses symm. ultrametric theory



Every 4-set does but not the entire 5-set

$A \in \binom{X}{2} \Rightarrow D(A) = 0$

$A \in \binom{X}{3}$ $D(A) = 1$ if not in A config, else 0.