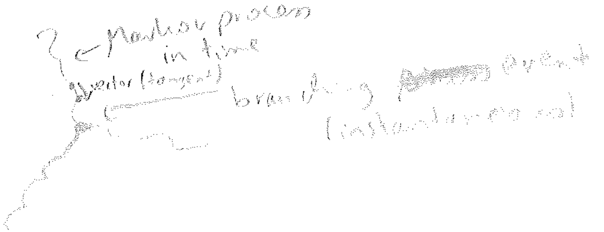
Jeremy's talk (Corsica 2012)



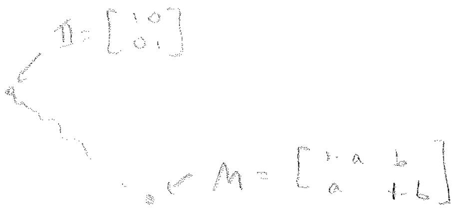
Markov process

State space $\{0,1\}$

Markov/transition/substitution matrix:

$$M = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} \end{array}$$

$a \equiv$ (prob. of transition from $0 \to 1$)



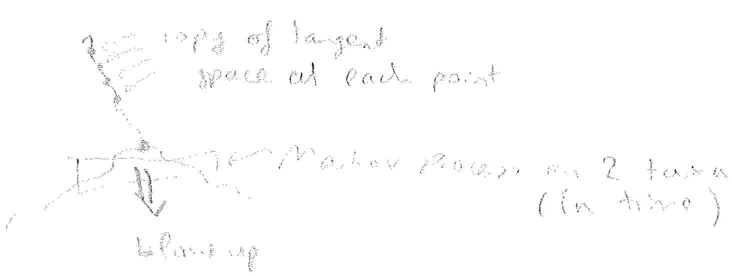
(Bemerkung: a & b sind eigentlich $a(t)$ & $b(t)$, also Funktionen, die von der Zeit t abhängen.)

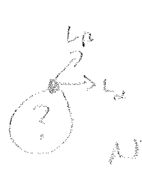
Tangents are then limits $a,b \to 0$ (i.e. wir gehen zurück auf dem Branch bis zu I)

$$L_\alpha := \frac{\partial M}{\partial a}\bigg|_{a=b=0} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \qquad L_\beta := \frac{\partial M}{\partial b}\bigg|_{a=b=0} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

Tangent space is a <u>vector space</u>, i.e. $Q = \alpha L_\alpha + \beta L_\beta = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}$ i.e. rate matrices!

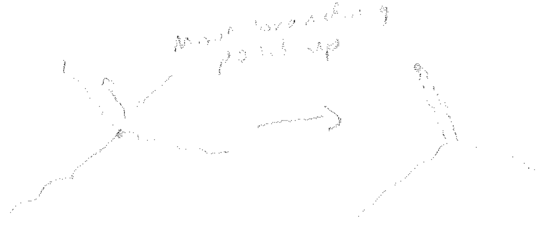
$\alpha, \beta \geq 0$ (aber: wegen Vector space eigentlich real arbitrary numbers!)



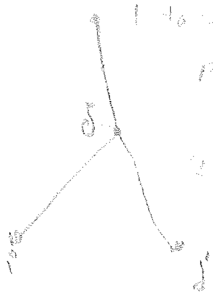


What happens after branching event, where Markov process is now independent across branches.

What this leads to is:



What about the branching event?
What is the tree describing?



1 taxon
prob. distr. $p \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}$ Just Before

2 taxa
prob. distr. $P = [p_{ij}]_{0 \le i,j \le 1} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}$
(Just after)

Branching event sends $p \to P$.

What is a linear operator δ that sends $p \to P$.

What do we know? Well, $p_{ij} = \begin{cases} p_i & \text{if } i = j \\ 0 & \text{else} \end{cases}$

$p = \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = p_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + p_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{C}^2$

$= p_0|0\rangle + p_1|1\rangle$

~~Kronecker prod.~~ Tensor prod.

and $P = p_{00}|0\rangle \otimes |0\rangle + p_{01}|0\rangle \otimes |1\rangle$
$+ p_{10}|1\rangle \otimes |0\rangle + p_{11}|1\rangle \otimes |1\rangle$

Notation:
$e_0 = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$e_1 = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$|00\rangle := |0\rangle \otimes |0\rangle$
$|01\rangle := |0\rangle \otimes |1\rangle$
$|11\rangle := |1\rangle \otimes |1\rangle$
$P \in \mathbb{C}^2 \otimes \mathbb{C}^2$

What about "branching operator"?
We need $\delta : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$
However, check that $\delta(|i\rangle) = |i\rangle \otimes |i\rangle = |ii\rangle$
does the right thing, i.e. $p \to \delta(p)$ ✓

Let's do it: $\delta(p) = \delta(p_0|0\rangle + p_1|1\rangle)$
$= p_0\,\delta(|0\rangle) + p_1\,\delta(|1\rangle)$
$= p_0|00\rangle + p_1|11\rangle$ ✓

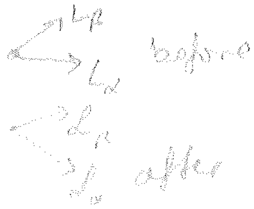
δ is defined with regard to the chosen basis ⇒ enough because linear!

[illegible]

$u_1 \otimes v_1 + u_2 \otimes v_2$
$u_1 \otimes (v_1 + v_2) = u_1 \otimes v_1 + u_1 \otimes v_2$
$u_1 \otimes (\lambda v_1) = \lambda (u_1 \otimes v_1)$
$A, B : \mathbb{C}^2 \to \mathbb{C}^2$
$v_1 = A v_1 \quad v_2 = B v_2$

$A \otimes B : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$
$A \otimes B : u \otimes v := (Au) \otimes (Bv)$

Corresponds to branching event: $\rho \to M_1 \otimes M_2 \cdot \delta(\rho)$



tangent space

Before branching: ~~$\delta L \ldots (\rho)$~~ $\delta L_\alpha\ (\rho)$

~~After branching~~ Intertwining: ~~$\delta L_\alpha \ldots (\rho)$~~ $\mathcal{L}_\alpha\, \delta\ (\rho)$

Want: ~~$\delta L_\alpha \ldots$~~ $\delta L_\alpha = \mathcal{L}_\alpha\, \delta$

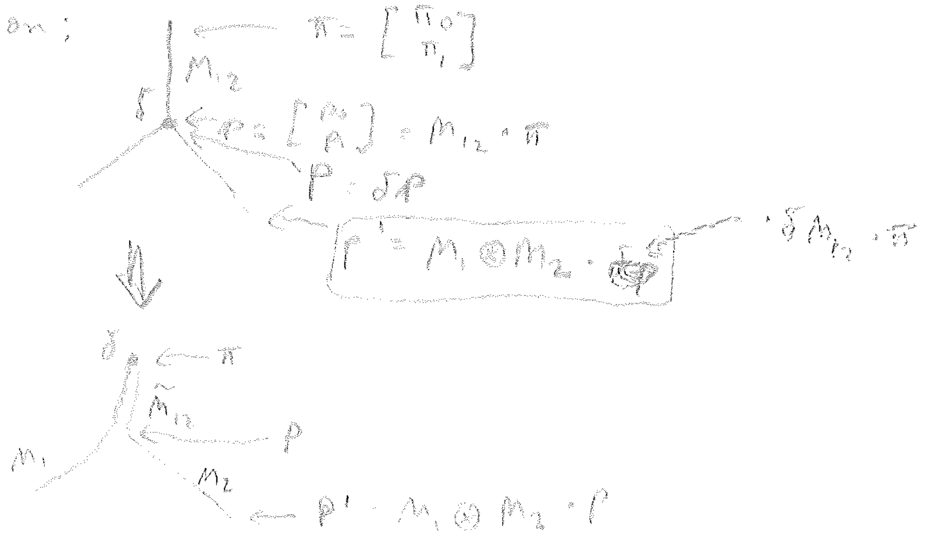
One can check: As operators from $\mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$,

$$(L_\alpha \otimes L_\alpha + L_\alpha \otimes 1 + 1 \otimes L_\alpha) \cdot \delta = \delta \cdot L_\alpha$$

i.e. $$\boxed{\mathcal{L}_\alpha = L_\alpha \otimes L_\alpha + L_\alpha \otimes 1 + 1 \otimes L_\alpha}$$

So now let's get out of the "blow up".

Standard description:



If we take $M_\tau = e^{Qt}$ $\quad Q = \alpha L_\alpha + \beta L_\beta$

$$= 1 + \sum_{n=1}^{\infty} \frac{(\alpha L_\alpha + \beta L_\beta)^n t^n}{n!}$$

$=$ Blah blah

Homework: Check $\{\mathcal{L}_\alpha, \mathcal{L}_\beta\}$ satisfy same algebraic relations as $\{L_\alpha, L_\beta\}$,

$$\text{i.e. } L_\alpha^2 = -L_\alpha, \quad L_\beta^2 = -L_\beta$$

$$L_\alpha L_\beta = \begin{bmatrix} \cdot & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} = -L_\beta$$

$$L_\beta L_\alpha = -L_\alpha$$

Check: $\mathcal{L}_\alpha^2 = -\mathcal{L}_\alpha$, $\mathcal{L}_\beta^2 = -\mathcal{L}_\beta$, $\mathcal{L}_\alpha \mathcal{L}_\beta = -\mathcal{L}_\beta$, $\mathcal{L}_\beta \mathcal{L}_\alpha = -\mathcal{L}_\alpha$

We have ~~$\tilde{M}_\tau \, \mathcal{F}$~~ $\tilde{M}_\tau \mathcal{F} = \mathcal{F} M_\tau$

$$= \mathcal{F}\left(1 + \sum_{n=1}^{\infty} \frac{(\alpha L_\alpha + \beta L_\beta)^n t^n}{n!}\right)$$

$$= \left(1 \otimes 1 + \sum_{n=1}^{\infty} \frac{(\alpha \mathcal{L}_\alpha + \beta \mathcal{L}_\beta)^n t^n}{n!}\right)\mathcal{F}$$

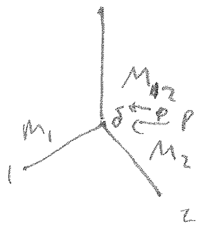
$$= e^{(\alpha \mathcal{L}_\alpha + \beta \mathcal{L}_\beta)t} \, \mathcal{F}$$

Ref: David Bryant; the ... paper (Hadamard),

but this works for group based models, whereas Jeremy's approach works for all

Jeremy's Talk Part 2 Cars 2017

This morning:



$M = e^{Qt}$

$Q = \alpha L_\alpha + \beta L_\beta$

$= \alpha \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$

$\delta \cdot L_\alpha \{\rho\}$:

$$L_\alpha = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & -1 & 0 \\ 1 & 1 & 0 \end{array}$$

$\delta L_\alpha = \mathcal{L}_\alpha \cdot \delta = (L_\alpha \otimes L_\alpha + \mathbb{1} \otimes L_\alpha + L_\alpha \otimes \mathbb{1})$

$\delta \cdot L_\alpha : \mathbb{C}^2 \to \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$

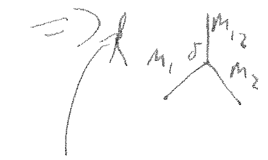
$\mathcal{L}_\alpha \cdot \delta : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$

$$\delta M_{12} = \delta e^{Q_{12} t} = \delta e^{(\alpha_{12} L_\alpha + \beta_{12} L_\beta) t}$$

$$= \delta \left(\mathbb{1} + \sum_{n=1}^{\infty} \frac{Q_{12}^n t^n}{n!} \right)$$

$$= e^{(\alpha_{12} \mathcal{L}_\alpha + \beta_{12} \mathcal{L}_\beta) t} \cdot \delta = \tilde{M}_{12} \cdot \delta$$

$\Rightarrow$



$\equiv$

} along here, we now have two binary ~~DNA~~ sequences, but they remain identical! After bifurcation independent development.

here, just one sequence and suddenly two from bifurcation on

don't matter as sequences are always identical in our case!

each state has two letters

$\tilde{M}_{12}$:

	0 0	0 1	1 0	1 1
0 0	-1	0	0	0
0 1	0	-1	0	0
1 0	0	0	-1	0
1 1	1	1	1	0

Recall: $\tilde{M}_{12} = e^{(\alpha_{12} d_\alpha + \beta_{12} d_\beta)t}$ in previous example

So: Now $\tilde{M}_{23} = e^{(\alpha_{23} d_\alpha + \beta_{23} d_\beta)t}$

$$\tilde{M}_{123} = e^{(\alpha_{123} d_\alpha^{(123)} + \beta_{123} d_\beta^{(123)})t}$$

$$d_\alpha^{(123)} = L_\alpha \otimes L_\alpha \otimes L_\alpha + L_\alpha \otimes L_\alpha \otimes 1 + L_\alpha \otimes 1 \otimes L_\alpha + 1 \otimes L_\alpha \otimes L_\alpha + L_\alpha \otimes 1 \otimes 1 + 1 \otimes L_\alpha \otimes 1 + 1 \otimes 1 \otimes L_\alpha$$

terms $= 2^3 - 1$

$$= \sum_{A \subseteq \{1,2,3\}, A \neq \emptyset} L_\alpha^{(A)}$$

(Remember: $d_\alpha = L_\alpha \otimes L_\alpha + L_\alpha \otimes 1 + 1 \otimes L_\alpha$
$= L_\alpha^{\{1,2\}} + L_\alpha^{\{1\}} + L_\alpha^{\{2\}}$)

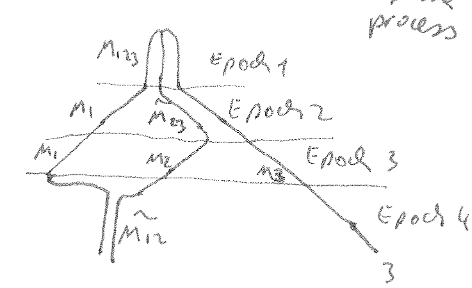
$\delta^2 \pi \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, so $[\delta^2 \pi]_{ijk} = \begin{cases} \pi_0, & \text{if } i=j=k=0 \\ \pi_1 & \text{if } i=j=k=1 \\ 0 & \text{else} \end{cases}$

$$\delta^2 \pi \xrightarrow{\text{Epoch 1}} \tilde{M}_{123} \cdot \delta^2 \pi$$

$$\xrightarrow{\text{Epoch 2}} M_1 \otimes \tilde{M}_{23} \cdot \tilde{M}_{123} \cdot \delta^2 \pi$$

$$\xrightarrow{\text{Epoch 3}} \underbrace{M_1 \otimes M_2 \otimes M_3} \cdot \underbrace{M_1 \otimes \tilde{M}_{23}} \cdot \underbrace{\tilde{M}_{123}} \cdot \delta^2 \pi$$

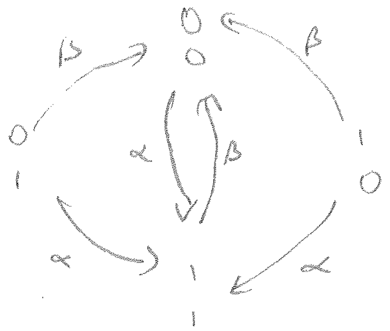
Now assume in Epoch 4 taxa 1 & 2 have the same process



↪ Nicht mehr treelike
& plötzlich andere States in d_α erreichbar,
da gleicher Prozess auf 1 & 2, aber
nicht mehr identische Sequenzen

$$
\mathcal{L}_\alpha = \begin{array}{c|cccc}
 & {}^{0}_{0} & {}^{0}_{1} & {}^{1}_{0} & {}^{1}_{1} \\
\hline
{}^{0}_{0} & -1 & 0 & 0 & 0 \\
{}^{0}_{1} & 0 & -1 & 0 & 0 \\
{}^{1}_{0} & 0 & 0 & -1 & 0 \\
{}^{1}_{1} & 1 & 1 & 1 & 0
\end{array}
$$

In the limit, 1 & 2 will be identical again



→ because nothing sends them out of the identical states in the Markov diagram!