Differentiating x^n in a Ring

Primer Talk, University of Canterbury

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Observation: The product rule of differentiation can be derived using the two rules

$$\begin{aligned} D\,(u+v) &= Du + Dv, \\ D(u^2) &= 2u\,Du. \end{aligned}$$

For then,

$$\begin{aligned} 2(u+v)D(u+v) &= D\left((u+v)^2\right) \\ &= D\left(u^2 + 2uv + v^2\right) \\ &= 2u\,Du + D\,(uv) + 2v\,Dv. \end{aligned}$$

Now expand the left side and solve for $D(uv)$.

Lift this to the context of ring theory.

Let R be a ring (not necessarily commutative). A *derivation* on R is an additive mapping $D : R \longrightarrow R$ such that

$$D(xy) = x\,Dy + (Dx)\,y \qquad (x, y \in R)\,.$$

Such a map has the nth *power property*: for each integer $n \geqslant 2$ and each $x \in R$,

$$D\left(x^n\right) = \sum_{k=1}^{n} x^{n-k}\,(Dx)\,x^{k-1}.$$

In the commutative case, this reduces to

$$D\left(x^n\right) = nx^{n-1}\,Dx.$$

Question: under what conditions is an additive mapping D on R that satisfies the nth power property a derivation?

Lemma 1 *If R is a commutative ring with no 2-torsion—that is,*

$$\forall_{x \in R} (2x = 0 \Rightarrow x = 0)$$

—then every additive map $D : R \to R$ with the 2nd power property is a derivation.

A nonzero ring R is *prime* if for any $a, b \in R$ such that $axb = 0$ for all $x \in R$, we have $a = 0$ or $b = 0$.

If R has a multiplicative identity e, then the *characteristic* of R is the smallest positive integer n such that

$$\underbrace{e + e + \cdots + e}_{n \text{ terms}} = 0$$

if such n exists; otherwise, the characteristic is 0.

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Theorem (Herstein, 1957) *Let R be a prime ring that is not of characteristic 2, and let $D : R \to R$ be an additive map with the 2nd power property. Then D is a derivation on R.*

Observation: For a fixed integer $n \geqslant 2$, let A be the $(n-1)$-by-$(n-1)$ matrix with (k, j)th entry

$$\binom{n}{j} k^j .$$

Then

$$\delta \equiv \det A = (n-1)! \left(\prod_{k=1}^{n-1} \binom{n}{k} \right) \prod_{1 \leqslant i < j \leqslant n-1} (i-j) .$$

This is needed for the proof of:

Lemma 2 *Let R be a ring, $n \geqslant 2$ an integer, $D : R \longrightarrow R$ an additive map with the nth power property, and y an element of R such that both y and Dy are in the centre of R. Suppose that for each $x \in R$, if $\delta x = 0$, then $x = 0$. Then*

$$D\left(x^k y^{n-k}\right) = y^{n-k} \sum_{j=1}^{k} x^{k-j} (Dx) x^{j-1} + (n-k) x^k y^{n-k-1} Dy.$$

Proof.

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Proof. Prerequisite: several beers.

Main Theorem (Bridges and Bergen, 1984) *Let R be a ring, $n \geqslant 2$ an integer, and $D : R \to R$ an additive mapping with the nth power property. Then D is a derivation under each of the following circumstances:*

(i) *R is commutative with an identity e, and for each $x \in R$, if $\delta x = 0$, then $x = 0$.*

(ii) *R is a prime ring with an identity e and characteristic c, where $c = 0$ or $c > n$.*

(iii) *R is a commutative integral domain* with characteristic c, where $c = 0$ or $c > n$.*

*That is, ring without right or left zero-divisors.

Proof for case (i). Since

$$De = D\left(e^n\right) = ne^{n-1}\, D\left(e^{n-1}\right) = n\, De,$$

we have $(n-1)\ De = 0$.

Hence $\delta\, De = 0$ and therefore $De = 0$.

Recall the conclusion of Lemma 2:

$$D\left(x^k y^{n-k}\right) = y^{n-k} \sum_{j=1}^{k} x^{k-j}\,(Dx)\,x^{j-1} + (n-k)\,x^k y^{n-k-1}\,Dy.$$

Take $y = e$ and $k = 2$, to get $D\left(x^2\right) = 2x\,Dx$.

Hence result, by Lemma 1. ◀