Informal seminar

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Finite time ergodic theory: nonsense, nuisance or needed?

Plan
1. Dynamical systems (finite time ≡ nonsense)
2. Open systems (finite time ≡ nuisance)
3. Finite data / non-autonomous (finite time ≡ needed)

1. Dynamical systems

Setup: $T : X \to X$ — $(X, \mathcal{S})$ measurable space → compact metric space (for simplicity)

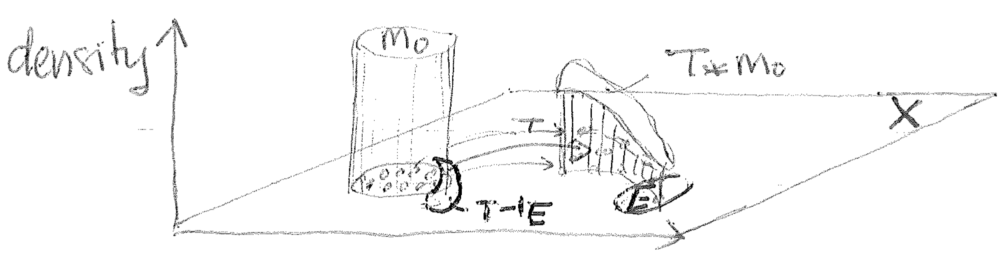
Convenient to have T continuous, and $\mathcal{S}$ the Borel σ-algebra. Also convenient to assume an ambient measure m [eg. Lebesgue if $X \subseteq \mathbb{R}^d$]

Remark: T may be "given" or arise as a "time-τ" map solving an autonomous ODE.

Orbits of $x \in X$ are obtained by iterative application of T: $\mathcal{O}^+(x) = \{x, T(x), T(T(x)), \ldots, T^n(x), \ldots\}$
→ $+$ denotes "future" → T^n: n-fold composition

These dynamics induce a flow on measures.

Eg: let m_0 be "uniform" on a small set, and use T to map points distributed like m_0



As $x \mapsto T(x)$, $m_0 \mapsto T_* m_0 := m_0 \circ T^{-1}$
(The value of $T_* m_0(E)$ is determined by measuring $T^{-1}E$ by m_0.)

The basic objects of ergodic theory are <u>invariant measures</u> : $\mu = T_*\mu$.

They are important because if μ is an invariant probability measure ($\mu = \mu\circ T^{-1}$, $\mu(X) = 1$)....

<u>Birkhoff's Ergodic Theorem</u> : if $f : X \to \mathbb{R}$ and $\int_X |f| d\mu < \infty$ then

$$\frac{1}{n}\left(f(x) + f\circ T(x) + \cdots + f\circ T^{n-1}(x)\right) \longrightarrow \bar{f}(x)$$

as $n \to \infty$ where convergence is for μ-almost every x (and is in L^1) and $\bar{f}$ is an <u>invariant</u> function ($\bar{f} = \bar{f}\circ T$ almost everywhere).

<u>Example</u> : E a measurable set, $f = \mathbb{1}_E$.
Then $f(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$
LHS in ergodic theorem is "time average" of f, so = proportion of time $O^+(x)$ spends in E.

If T is <u>ergodic</u> then $\bar{f}$ is a <u>constant function</u>. Since $\int_X \bar{f} d\mu = \int_X f d\mu$, ergodicity is interpreted as :

"time averages" = "space averages"
$[\bar{f}(x) = \int_X f d\mu$ a.e.$]$

<u>Historical connection</u> : Hamiltonian systems preserve phase space volume. Not ergodic as orbits confined to surfaces of constant energy. For complicated interacting particle systems it is often hoped that there is "ergodicity on energy surfaces" — frequently FALSE!

<u>In general</u> : $\bar{f}$ obtained as time averages of integrable f are "constant on ergodic components". ? Can these be resolved computationally ?

Important remarks on Birkhoff's Theorem

* an asymptotic result; silent on rates, which can be arbitrarily bad, depending on regularity of f (and T)

* an almost everywhere result – so strength of statement depends on μ [eg if $x = Tx$) then the "point mass" δ_x is invariant, but x is the only typical point].

The first point suggests finite time ergodic theory = nonsense. The second point has asymptotic issues buried in it too:

Given T, are there invariant measures? Yes. Start with any m_0 and compute

$$m_n = \frac{1}{n}\left(m_0 + T_* m_0 + \cdots + (T_*)^{n-1} m_0\right).$$

By Alaoglu's Theorem, $\{m_n\}_{n=0}^{\infty}$ has a weak-* limit point. It is easily verified to be invariant.

– again, an asymptotic construction!

And also only a partial answer to the behaviour of typical orbits. We'd like:

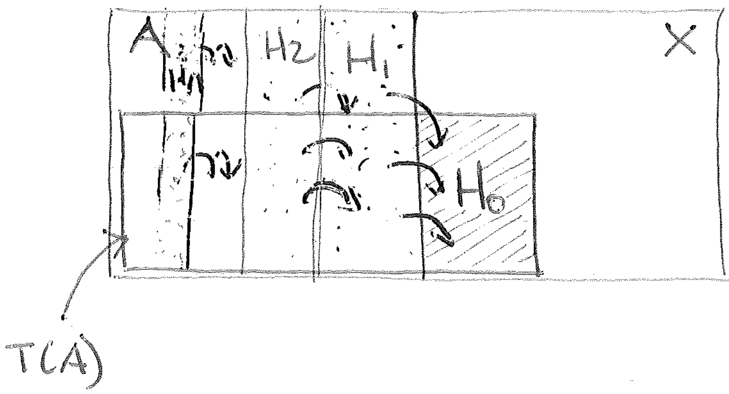
Natural measure μ_*: if m_0 is any reasonable initial measure (eg absolutely continuous wrt m = "has density function") then

$$(T_*)^n m_0 \longrightarrow \mu_* \quad \text{as} \quad n \to \infty.$$

Ideally, with rates of convergence. Sometimes this is possible (certain "uniformly expanding" T), but in general "ergodic behaviour" is beyond grasp of "finite iteration".

(2.) Open systems

We keep most of structure of (1), but turn attention to a subset $A \subsetneq X$ from which *escape* is possible: $T(A) \not\subset A$



Now $H_0 = T(A) \setminus A$ is a *hole* through which orbits escape from A. Let

$$\begin{aligned} H_1 &= A \cap T^{-1} H_0 \\ H_2 &= A \cap T^{-1} H_1 \\ &\vdots \\ H_n &= A \cap T^{-1} H_{n-1} \\ &\vdots \end{aligned}$$

It could be that $m(\overbrace{A \setminus (\bigcup_{n=1}^{\infty} H_n)}^{\text{"survivor set"}}) = 0$, so *most* orbits escape from A in finite time.

We'd like a measure which captures *escape statistics*.

- invariant measure may be too much to hope for (supported on ~~...~~ trivial "survivor set")

- "quasi-invariant measure": $T_* \mu = \rho \mu$ for some $\rho \in (0,1)$.

Problem: which μ is "natural" given that there are uncountably many such μ for each ρ. [Arbitrary distribution on H_1; pull back by T^{-1} to H_2 and scale by ρ; etc; normalise].

This last fact (many natural quasi-invariant measures) shows that "finite time" issues are at least a nuisance. Why are they important?

③ <u>The need for "finite time ergodic theory"</u>

(or at least, new approaches)

- sometimes you want to learn something about a problem with a <u>finite amount of computation or data</u> (can you do "statistics"?)
- sometimes a "finite" question is the one you want to answer (escape from open system).

Most arguments in proofs of results described here rely in some way on <u>compactness</u>. A lot of thinking is needed to extend these ideas to the (realistic) situation where dynamics is non-autonomous (~~time~~ dependent).

<u>Eg</u>: in dynamical systems we often handle time dependent ODEs by extending phase space $X \longmapsto X \times \mathbb{R}$ so

$$\left.\begin{aligned} \dot{x} &= f(x,t) \\ \dot{t} &= 1 \end{aligned}\right\} \text{autonomous}.$$

[clearly, $X \times \mathbb{R}$ non-compact].

Alternatively, $X \times$ [very large time interval] may be compact, but becomes open!

<u>Afterword</u>: the topic was motivated by the "what am I doing here" theme suggested by DSB. That topic was a lead-in to joint work with Chris Bose on Maximum Entropy selection of "natural" quasi-invariant measures.