Piṅgala's Fountain

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ABSTRACT

Piṅgala is a mathematician (c. 300 BCE) who authored *Chandaḥśāstra* (the science of Sanskrit metres). This is a brief note on a mathematical art project completed at the Chennai Mathematical Institute to honour Piṅgala who was responsible for the discovery of binary expansion of numbers in the context of enumeration of metres, and of the binomial coefficients which were later made explicit by Virahāṅka (c. 600 CE) and Halāyudha (c. 950 CE).

Keywords: Mathematical art, Binary numbers, binomial coefficient, Pascal's triangle, Fibonacci numbers

1. INTRODUCTION

Piṅgala's Fountain, the mathematical sculpture and water fountain, was designed, constructed and erected at the Chennai Mathematical Institute, Chennai, India in early December 2010. In this note we describe the design and the significance of the symbols carved into Piṅgala's Fountain. See Sainudiin (2010b) for more details and a video (Sainudiin, 2010a) of the fountain in action. Piṅgala (circa 300 BCE) discussed the theoretical basis of Sanskrit Prosody in his book *Chandaḥśāstra*. The book is written in the form of *sūtras* (pithy mnemonic rules). A metre is a finite sequence of two kinds of syllables called *laghu* (light) and *guru* (heavy), which are the basic units of speech. The length of a metre is the number of syllables occurring in it. The last chapter of *Chandaḥśāstra* contains sixteen sūtras which describe various combinatorial tools (called *pratyayas* by later commentators) which arise in the mathematical investigation of metres. The first among them is *prastāra* which is an explicit rule to enumerate all metres of length n in the form of an array. The array begins with the metre of a given number of syllables which are

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all gurus and ends with the metre having all laghus, constructed by an inductive process. The next tool, *saṇkhyā*, gives a method to calculate the total number (2^n) of metres of length n. Two tools, *naṣṭa* and *uddiṣṭa*, are introduced in order to set up a one-to-one correspondence between each metre of length n and the number of the row in which it occurs, with the convention that the first row corresponds to the number zero. This arrangement corresponds to the binary expansion of any integer between 0 and $2^n - 1$.

Piṅgala also considers the delicate problem of determining the number of metres in the *prastāra* of metres of length n, which consist of r gurus, which in modern parlance is the binomial coefficient $\binom{n}{r}$. This is achieved by Piṅgala by

Figure 1. Photograph of the sculpture Piṅgala 's Fountain at the Chennai Mathematical Institute

constructing a *meru* (a mountain of numbers), which he describes by a particularly cryptic rule: *Pare pūrṇamiti*. This rule which is inscribed in Sanskrit at the top of our sculpture indeed leads to the construction of the so called Pascal's triangle, as has been explained by the commentators on Piṅgala's work such as Halāyudha (circa 10th century) and others. As Halāyudha explains, "*Pare pūrṇam*" in the rule should be taken to mean that each number is obtained by adding together the numbers that occur above it in the previous row. This of course is the well known recursion relation $\binom{n}{r} = \binom{n}{r-1} + \binom{n-1}{r-1}$. The numbers in the n-th row give the number of metres of length $n-1$ with different possible numbers of gurus and they add up to 2^{n-1}, which is the total number of metres of length $n-1$. See Sridharan (2005) for more details on the above relations by Piṅgala between integers, binary numbers and binomial coefficients.

It is of interest to note that this *meru* also occurs, now in a "tilted" form, in the study of the so called *mātra* or moric metres. The light syllable *laghu* or L is equal to one mora and the heavy syllable guru or G is equal to two morae. There is one possible variation (L) for a one-mora metre and two possible variations (G and LL) for a two-morae metre. By appending G to the only one-mora metre and appending L to each of the two two-morae metres on the right we get three possible variations (LG, GL and LLL) of the three-morae metre. Similarly, by appending G to each of the two two-morae metres and appending L to each of the three three-morae metres on the right we get the five possibilities (GG, LLG, LGL, GLL and LLLL) of the four-morae metre. Thus, the *saṅkhyā* or the total number of metres of n-morae turns out to be F_n, which is the n-th Fibonacci number. This leads to the interesting property that the entries of the *meru* (or the Pascal's triangle), when summed along the "diagonals", yield the Fibonacci numbers (Sridharan, 2006; Sridharan et al., 2012). These numbers are carved along the right diagonal in our sculpture. A recent book (Kusuba and Plofker, 2013, p. 47) features the fountain with the following words:

> A fountain at the Chennai Mathematical Institute commemorates the combinatorial algorithms of the ancient Indian scholar Piṅgala's. It features numbers in what is now called the Fibonacci sequence.

2. DESIGN AND CONSTRUCTION

The design and construction was a collaboration between mathematicians, artists, and local masons and inspired by Galton's Quincunx (Galton, 1889, p. 62–65).

Piṅgala's fountain is made of two pieces of carved granite rock. The bottom part of the fountain forms a heavy pedestal upon which the top part is locked and afixed by gravity. The pedestal is a rare piece of antique rock that had been quarried by hand into a cuboidal shape. The upper piece of the fountain was carved by more modern machine tools and mounted onto the pedestal. A basin to hold and recycle the water was carved into the pedestal.

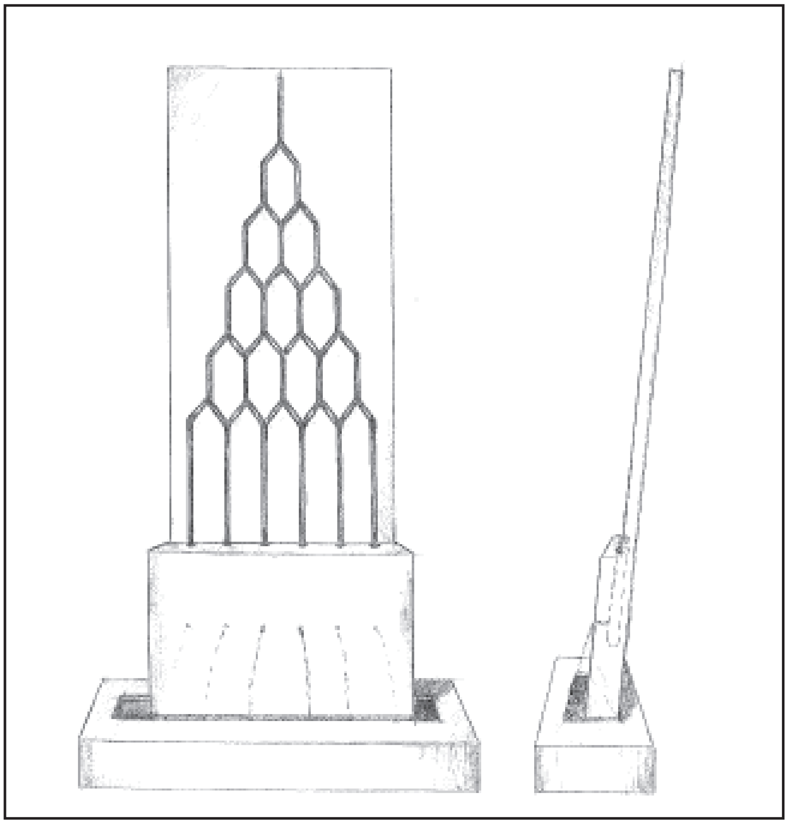

Figure 2: Artist's sketch

Water is pumped from the basin to the top of the fountain where it gently cascades through a depressed channel which bifurcates at intervals to eventually flow through six channels as illustrated in Figure 2. The water then collects in large cylindrical wells inside the granite base with vents to the outside of the slab through small perpendicularly positioned channels. This produces a fountain effect

Figure 3: Building Piṅgala's Fountain in Mamallapuram. Sculptors (bottom image from left to right): V. Rajagopal, J. Paul Victor, V. Kannan, K. Ravichandran and R. Dhanasekaran. Two more sculptors, Thanikasalam and Doss, are not in the picture.

with water flowing back into the basin at different pressures depending on the volume of water collecting in each of the six leaky wells. Naturally, due to the binomal branching and merging of the channels within the Hexagonal lattice, more water collects in the middle two wells compared to the wells at the two outer ends. After several unsuccessful attempts with other geometries for water channels we finally settled on the hexagonal lattice to embed the binomial branching and merging of channels to carry water down and accumulate within the leaky wells in proportion to their binomial coefficients (up to statistical fluctuations).

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REFERENCES

[1] F. Galton. Natural inheritance. Macmillan and Co., 1889.

[2] T. Kusuba and K. Plofker. Indian combinatorics. In: R. Wilson and J. Watkins, eds., *Combinatorics*: *Ancient and Modern*, 41−64. Oxford University Press, Oxford, UK, 2013.

[3] R. Sainudiin. Piṅgala's Fountain in Action. http://www.youtube.com/watch?feature=player_detailpage&v=qHhSbJSTDLk, 2010a. [Online; accessed 14-December-2013].

[4] R. Sainudiin. Piṅgala's Fountain, Laboratory for Mathematical Statistical Experiments, Christchurch Centre. http://www.math.canterbury.ac.nz/~r.sainudiin/lmse/pingalas-fountain/, 2010b. [Online; accessed 14-December-2013].

[5] R. Sridharan. Sanskrit prosody, Piṅgala's sastra and binary arithmetic. In: G. Emch, M. Srinivas, and R. Sridharan, eds., *Contributions to the History of Mathematics in India,* 33−62. Hindustan Publishing Agency, Delhi, India, 2005.

[6] R. Sridharan. Pratyayas for màtràvç1ttas and fibonacci numbers. *Mathematics Teacher*, 42:120−137, 2006.

[7] R. Sridharan, R. Sridharan, and M. Srinivas. Differences in style but not in substance: Ancient indian study of combinatorial methods. In: R. Sujatha, BH. Ramaswamy, and C. Yogananda, eds., Math Unlimited: Essays in Mathematics, 423−440. CRC Press, Jersey, UK, 2012.

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