**UNIVERSITY OF CANTERBURY**

# MID YEAR EXAMINATIONS 2011

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| **Prescription Number(s):** | **STAT 313 - 11S1 (C)** |
| **Paper Title:** | **Computational Statistics** |

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| Time allowed: | TWO HOURS |
| Number of Pages: | 4 |

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**Read these instructions carefully**.

1. This is an open-book, open-notes take-home exam.
2. You may use a pocket calculator and/or a computer.
3. At the end of the exam, **hand in** the answer booklet(s) to Erskine Reception 4th floor.
4. Answer **all** questions. All questions carry the said weight.

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**Q.1** (Total **20%**) For a given ordered triple of parameters $(\theta_1, \theta_2, \theta_3) \in (-\infty, \infty)^3$, such that $\theta_1 < \theta_2 < \theta_3$, the random variable $X$ is said to have the triangular distribution and denoted $X \sim \mathrm{TRI}(\theta_1, \theta_2, \theta_3)$, if its PDF is:

$$f(x; \theta_1, \theta_2, \theta_3) = \begin{cases} \frac{2(x-\theta_1)}{(\theta_3-\theta_1)(\theta_2-\theta_1)} & \text{if } \theta_1 \leq x \leq \theta_2 \\ \frac{2(\theta_3-x)}{(\theta_3-\theta_1)(\theta_3-\theta_2)} & \text{if } \theta_2 \leq x \leq \theta_3 \\ 0 & \text{otherwise} \end{cases} .$$

Devise an algorithm (in pseudocode) that can transform IID samples from $U \sim \mathrm{Uniform}(0, 1)$ into IID samples from

$$X \sim \mathrm{TRI}(\theta_1, \theta_2, \theta_3) \ .$$

Give all details of your algorithm and show all steps in the derivation of any expressions used in the algorithm.

**Q.2** (Total **15%**) Let $X_1$ and $X_2$ be independent Poisson random variables with means $\mu_1$ and $\mu_2$, where $\mu_1 > \mu_2 \geq 0$.

(a. 5%) Devise an algorithm (in pseudocode) that can transform IID samples from $U \sim \mathrm{Uniform}(0, 1)$ into IID samples from $X_1$. Explain the ide or principle behing your algorithm.

(b. 5%) Prove or disprove: $X_1 + X_2$ has Poisson distribution with mean $\mu_1 + \mu_2$.

(c. 5%) Prove or disprove: $X_1 - X_2$ has Poisson distribution with mean $\mu_1 - \mu_2$.

**Q.3** (Total **10%**) A random variable $X$ is simulated using the following method:

Step 1. Generate two independent $\mathrm{Uniform}(0, 1)$ random variables $U$ and $V$.

Step 2. If $U^2 + V^2 \geq 1$, return to Step 1; otherwise set $X \leftarrow U$.

What is the distribution function of $X$? How many times will step 1 be performed (answer in terms of the mean and standard deviation)?

**Q.4** (Total **10%**) Suppose I drop a ball into the Quincunx and only observe that it ended up at the bucket labelled 7. The buckets are labelled by the number of right turns made by the ball along its journey down the 21 levels of nails.

(a. 2%) How many left turns did the ball make under the assumption that it behaved well, in terms of only going right or left at each nail it hit, along its journey?

(b. 2%) What is the probability of this observed outcome under a binomial model with 21 trials and equal probability of taking a right versus left turn, i.e. with parameters $n = 21$ and $\theta = 1/2$?

(c. 2%) What is the likelihood function of the parameter $\theta \in [0, 1]$ for this observation under the binomial model?

(d. 2%) If you had to choose the most likely parameter from the set

$$\{1/10, 1/2, 9/10\}$$

based on your single observation of 7 right turns, then what would it be? Justify your answer.

(e. 2%) If you dropped another ball and it fell into the bucket labelled 0 then what is your most likely estimate of $\theta$ from the same set above on the basis of this new ball's outcome alone? Justify your answer.

**Q.5** (Total **20%**) This question has two parts. Recall the Algorithm in pseudo-code for basic Monte Carlo integral estimation:

**Algorithm 1** Basic Monte Carlo Integral Estimation for $\vartheta^* = \int_{[\underline{a}_1,\overline{a}_1]\times\cdots\times[\underline{a}_k,\overline{a}_k]} h(x)dx$

*input:*

(a) $n \leftarrow$ the number of samples.

(b) $h(x) \leftarrow$ the integrand function over $\mathbb{R}$

(c) $[\underline{a}_j, \overline{a}_j] \leftarrow$ lower and upper bounds of integration for each $j = 1, 2, \ldots, k$

(d) capability to draw $nk$ IID samples from Uniform$(0, 1)$ RV

*output:* a point estimate $\widehat{\vartheta}_n$ of $\vartheta^*$ and the estimated standard error $\widehat{\mathsf{se}}_n$

*initialize:* $y \leftarrow (0, 0, \ldots, 0)$, initialize $y$ as a zero vector of length $n$

**while** $i \leq n$ **do**

(a) $i \leftarrow i + 1$,

(b) $x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,k})$, with $x_{i,j} \leftarrow u_j$, $u_j \sim$ Uniform$(\underline{a}_j, \overline{a}_j)$, for $j = 1, 2, \ldots, k$,

(c) $y_i \leftarrow w(x_i) = h(x_i) \prod_{j=1}^{k} (\overline{a}_j - \underline{a}_j)$

**end while**

(a) $\widehat{\vartheta}_n \leftarrow \overline{y}_n$, the sample mean of $y = (y_1, y_2, \ldots, y_n)$

(b) $\widehat{\mathsf{se}}_n = s_n(y)/\sqrt{n}$, where $s_n(y)$ is the sample standard deviation of $y$

*return:* $\widehat{\vartheta}_n$ and $\widehat{\mathsf{se}}_n$

(a. 10%) Imagine a rigid chunk of swiss-cheese $S$ that is perfectly enclosed by a hollow cube $C_3$. Suppose you have at your disposal the indicator function $\mathbb{1}_S((x_1, x_2, x_3))$ which returns 1 if the point $(x_1, x_2, x_3)$ is in $S$ and 0 otherwise. Describe in words **and** in pseudo-code how you would use Monte Carlo integral estimation to compute $\vartheta^*$, the volume of cheese in $S$.

(b. 10%) List the pseudo-code to compute the point estimate, standard error and a Normal-based 95% confidence interval for $\vartheta^*$. The listing should have details similar to Algorithm 1 above.

END OF PAPER