STAT221

Tutorial 10: Practice questions

1. Suppose three coins named C_1, C_2 and C_3 have probability 1/3, 1/2 and 2/3 of landing Heads, respectively. Suppose one of them is chosen and tossed five times in an IID manner with the following outcomes: `H, T, H, T, T`. What is your **most likely estimate** for the chosen coin?

   (A) Coin C_1 is most likely.

   (B) Coin C_2 is most likely.

   (C) Coin C_3 is most likely.

   (D) We cannot decide the most likely coin using the maximum likelihood principle.

2. Which one of the following algorithms can produce a sample from the RV X with DF:

$$F(x;\alpha) = 1 - \exp\left(\frac{-x^2}{2\alpha^2}\right) \quad ?$$

   (A) $u \sim \text{Uniform}(0,1)$; $\quad x \leftarrow \alpha\sqrt{-1/2\log(u)}$

   (B) $u \sim \text{Uniform}(0,1)$; $\quad x \leftarrow \alpha\sqrt{-2\log(u/2)}$

   (C) $u \sim \text{Uniform}(0,1)$; $\quad x \leftarrow \alpha\sqrt{-2\log(u)}$

   (D) $u \sim \text{Uniform}(0,1)$; $\quad x \leftarrow \alpha^2\sqrt{-2\log(u)}$

3. Suppose you observe the following five data points from some product experiment:

$$0, 2, 1, 3, 1, 0, 3, 1, 6.$$

   (a) draw a graph of the empirical mass function.

   (b) draw a graph of the empirical distribution function.

   We will not go through this in the tutorial, but you should also be able to:

   (a) compute the sample mean.

   (b) compute the sample variance.

   (c) compute the order statistics.

4. A linear congruential generator, $LCG(m, a, c, x_0, n)$ is given by:

$$x_i \leftarrow (ax_{i-1} + c) \mod m, \quad i = 1, 2, \ldots, n$$

(a) Show that the LCG with $(m, a, c, x_0, n) = (256, 138, 123, 13, 256)$ cannot have a full period.

(b) What could we change to make this LCG have a full period length of 256?

5. Given a real parameter $\lambda > 0$, the discrete RV X is said to be Poisson(λ) distributed if X has PDF:

$$f(x; \lambda) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & \text{if } x \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\} , \\ 0 & \text{otherwise} . \end{cases}$$

Now suppose we make n observations and model these as a product Poisson(λ^*) experiment:

$$X_1, X_2, \ldots, X_n \overset{IID}{\sim} \text{Poisson}(\lambda^*) \ .$$

(a) Set up the log-likelihood function $\ell(\lambda)$ for the observed data $(x_1, x_2, \ldots, x_n)$ and simplify the expression as much as you can.

(b) Find the derivative of $\ell(\lambda)$ with respect to λ.

(c) Solve for λ in the equation: $\frac{\partial \ell(\lambda)}{\partial \lambda} = 0$.

(d) What is the maximum likelihood estimator $\widehat{\Lambda}_n$ of the unknown parameter λ^*?

(e) If you observed 5 data points $(x_1, x_2, x_3, x_4, x_5) = (0, 2, 1, 0, 3)$, what is the maximum likelihood estimate $\widehat{\lambda}_5$ of λ^*?

6. Suppose a frog jumps between three lily pads labelled by the integers in $\{0, 1, 2\}$. If the frog is in lily pad i then it jumps to the lily pad $(i + 1) \mod 3$ with probability $3/5$ and to the lily pad $(i - 1) \mod 3$ with probability $1/5$.

(a) Draw the flow diagram for a Markov chain model of the frog jumps over the state space of the three labelled lily pads.

(b) Produce the 3×3 transition probability matrix $\mathbf{P}$ for the model.

(c) Compute the probability of being in state 2 in time-step 2 if you started from lily pad 1 at the initial time-step 0.

(d) Suppose you watched the frog make the following sequence of jumps:

$$0, 2, 0, 1, 1, 2, 0, 0, 1, 0, 1, 2, 0, 2, 2, 0, 1, 1, 2, 0, 1 \ .$$

What is the likelihood of the above data under the model with the given $\mathbf{P}$?

(e) Describe in words and in pseudo-code an algorithm that can produce sample paths (sequence of jumps) for this Markov chain with transition probability matrix $\mathbf{P}$ if you started at time 0 according to the initial probability vector $\mathbf{p}^{(0)} = (1/3, 1/3, 1/3)$ over the state space $\{0, 1, 2\}$.