Limits of Sequence of Real Numbers

Defn: A sequence of real numbers $\{x_i\}_{i=1}^{\infty} := x_1, x_2, \ldots$ is said to converge to a limit $a \in \mathbb{R}$:

$$\lim_{i \to \infty} x_i = a$$

if for every natural number $m \in \mathbb{N}$, a natural number $N_m \in \mathbb{N}$ exists such that for every $j \geq N_m$, $|x_j - a| \leq \frac{1}{m}$.

eg: $\{x_i\}_{i=1}^{\infty} = 17, 17, 17, \ldots$

For every $m \in \mathbb{N}$, take $N_m = 1$, then

$\forall j \geq N_m = 1$, $|x_j - 17| = |17 - 17| = 0 \leq \frac{1}{m}$. as needed.

for every.

eg:- $\{x_i\}_{i=1}^{\infty} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots$, i.e., $x_i = \frac{1}{i}$

then $\lim_{i \to \infty} x_i = \lim_{i \to \infty} \frac{1}{i} = 0$

For every $m \in \mathbb{N}$, take $N_m = m$, then

$\forall j \geq N_m = m$, $|x_j - 0| \leq \left|\frac{1}{m} - 0\right| = \frac{1}{m}$ as needed.

This is a recall of limits from 100 level math/stats courses or from high school. Now, let's recall limits of functions.

Dfn Limits of functions

We say a function $f(x): \mathbb{R} \to \mathbb{R}$ has a limit $L \in \mathbb{R}$ as x approaches a:

$$\lim_{x \to a} f(x) = L$$

Provided $f(x)$ is arbitrarily close to L for all values of x that are sufficiently close to but not equal to a.

Ex: $f(x) = (1+x)^{1/x}$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e \approx 2.71828\ldots$$

eventhough $f(0) = (1+0)^{1/0}$ is undefined!

—— x ——

This is a prequel to the topic of interest to us:

Limit of a Sequence of Random Variables.

$$\lim_{i \to \infty} X_i = X$$

We will introduce two most elementary RVs next to develop the intuition for limits of RVs.

We want to be able to say things like:

" $\lim_{i \to \infty} X_i = X$ " in some sensible way.

limiting random variable.

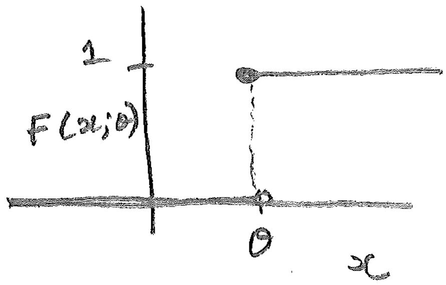
random variables

Model. Point Mass (θ) RV.

Given a specific point $\theta \in \mathbb{R}$, we say that X is a Point Mass (θ) RV if:

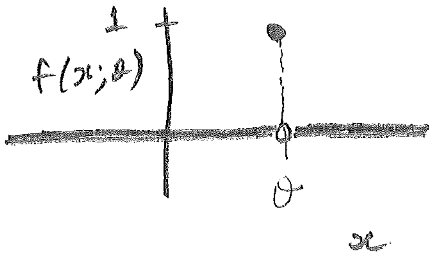
$$F(x;\theta) = \begin{cases} 0 & \text{if } x < \theta \\ 1 & \text{if } x \geq \theta \end{cases}$$

$$= \mathbb{1}_{[\theta,\infty)}(x)$$



$$f(x;\theta) = \begin{cases} 0 & \text{if } x \neq \theta \\ 1 & \text{if } x = \theta \end{cases}$$

$$= \mathbb{1}_{\{\theta\}}(x)$$



Remark:

The "variable θ" you have encountered in the non-random context is really the Point Mass (θ) RV X whose every realisation is θ !

Therefore, $\lim_{i \to \infty} \theta_i = \theta$ is also the same as

$\lim_{i \to \infty} X_i = X$, where $X_i \sim$ Point Mass(θ_i)

and $X \sim$ Point Mass (θ).

Now, we want to generalise this notion of limit to other sequence of RVs that are not necessarily Point Mass(θ_i) RVs.

Model Gaussian (μ, σ^2) or Normal (μ, σ^2) RV.

We say a RV X is Normal (μ, σ^2) if:

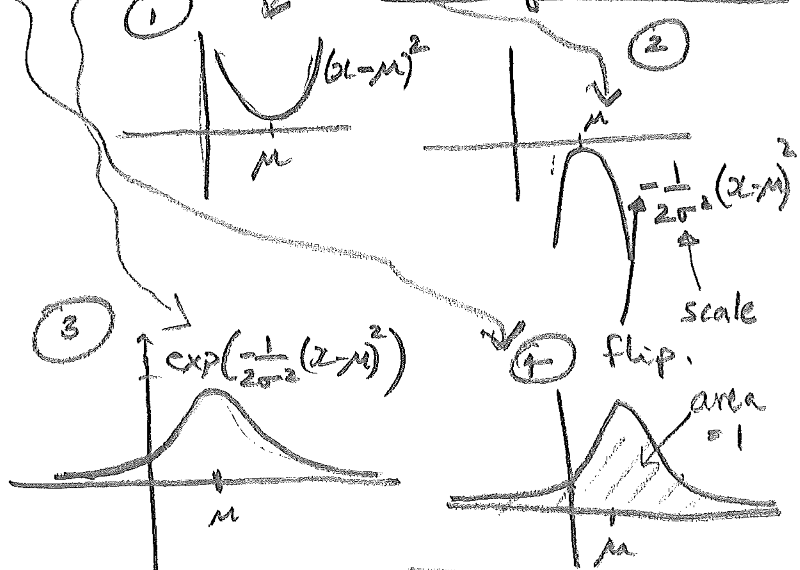
$$f(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(- \frac{1}{2\sigma^2} (x-\mu)^2 \right),$$

Visualize This in steps

where, $x \in \mathbb{R}$

Location Parameter $\rightsquigarrow \mu \in \mathbb{R}$.

scale Parameter $\rightsquigarrow \sigma^2 \in (0, \infty)$



$\boxed{\frac{1}{\sigma\sqrt{2\pi}}} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)$

normalizing constant so ▨ = 1

Now, imagine the effect of changing μ and σ^2 on the shape of the PDF. (SAGE interact;)

Exercise.

What does the shape of the PDF imply for realisations of the Normal (μ, σ^2) RV?

What happens when μ increases or decreases to the shape of the PDF? Similarly, what happens to PDF when σ^2 is increased or decreased?

Do you see why they are named location & scale parameters?

Consider the following Example:

Let $X_1 \sim \text{Normal}(0, 1)$

$X_2 \sim \text{Normal}(0, \frac{1}{2})$

$X_3 \sim \text{Normal}(0, \frac{1}{3})$

$\vdots$

$X_i \sim \text{Normal}(0, \frac{1}{i})$

$\vdots$

$X \sim \text{Point Mass}(0)$

The probability mass of X_i's increasingly concentrates about 0 as $i \to \infty$ and the variance $\frac{1}{i} \to 0$ for the Normal RV $X_i \sim \text{Normal}(0, \frac{1}{i})$

Based on this observation, we can surmise that

$X_i \to X \sim \text{Point Mass}(0)$?

NO is not true at all. because for any i, however large, $P(X_i \in \{0\} = 0)$ since X_i is a continuous RV.

Part of the problem is that we need to refine our notions of convergence when RVs are at play.

Dfn (Convergence in Distribution)

Let $X_1, X_2, \dots$ be a sequence of RVs and let X be another RV. Let F_n denote the D.F. of X_n and F denote the D.F. of X. Then, we say that X_n converges to X in distribution and write:

$$X_n \rightsquigarrow X$$

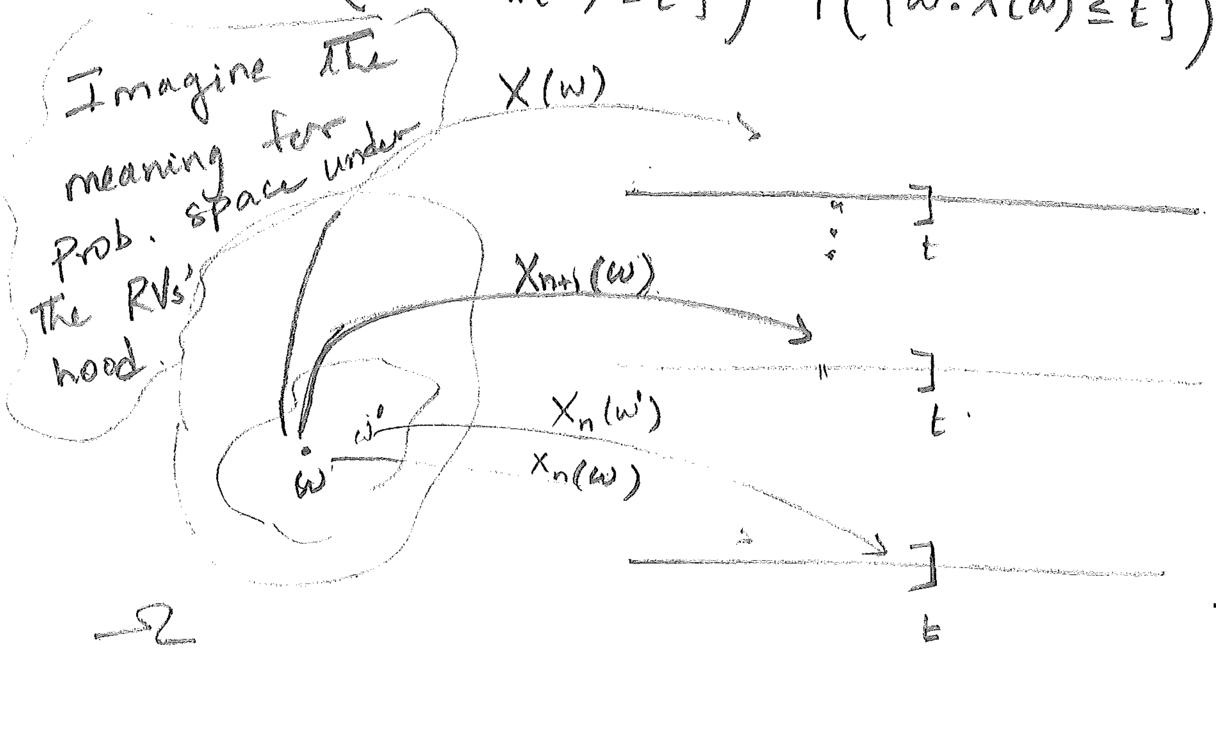
if for any real number t at which F is continuous,

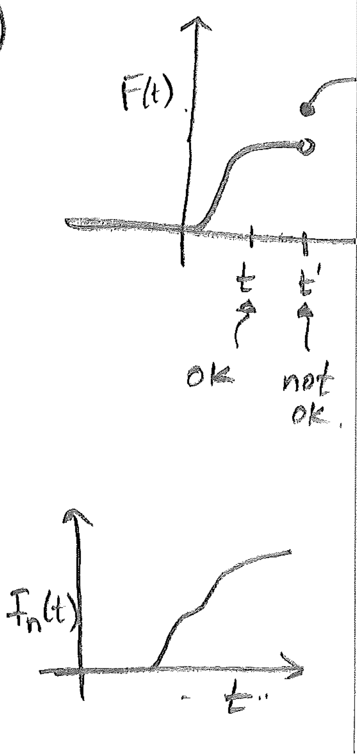
$$\lim_{n\to\infty} F_n(t) = F(t)$$

[in the sense of convergence or limits of functions (see Dfn Limits of Functions) from before.]

or equivalently:

$$\lim_{n\to\infty} P(\{\omega : X_n(\omega) \le t\}) = P(\{\omega : X(\omega) \le t\})$$





Dfn Convergence in Probability

Let $X_1, X_2, \dots$ be a sequence of RVs and let X be another RV. Let F_n denote the DF of X_n and F denote the DF of X. Then, we say X_n converges to X in probability and write:

$$X_n \xrightarrow{P} X$$

if for every real number $\varepsilon > 0$,

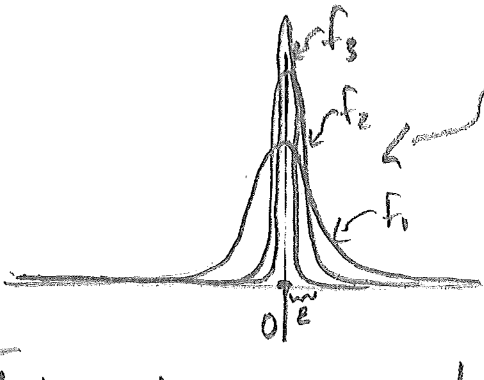
$$\lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0$$

equivalently,

$$\lim_{n \to \infty} P(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) = 0$$

Picture for Conv. in Prob.

If $X_n \sim \text{Normal}(0, \frac{1}{n})$ and $X \sim \text{PointMass}(0)$



complete the picture by working through the meaning of the Dfn.

so, $X_n \xrightarrow{P} X$

Also

Note: We can also say $X_n \rightsquigarrow X$ when $X_n \sim \text{Normal}(0, \frac{1}{n})$ and $X \sim \text{PointMass}(0)$.