Random Vectors.

Random Variables we have seen so far take real numbers as their realisations. Now, we want to look at random vectors whose realisations are real numbers in one or more dimensions.

Remark: Note that $X_1, \dots, X_n \overset{IID}{\sim} f$ we have been seeing can be viewed as a random vector $(R\vec{V})$ with product density i.e.,

$$(X_1, X_2, \dots, X_n) \sim \prod_{i=1}^{n} f(x_i).$$

Example:

Let X_1 and X_2 be IID Bernoulli$(\frac{1}{2})$ random variables (RVs)

	$X_2 = 0$	$X_2 = 1$	
$X_1 = 0$	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
$X_1 = 1$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
	$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$	$\frac{1}{2}$	

joint probabilities

Marginal probabilities

Model Bernoulli(θ_1, θ_2) Random Vector $(R\vec{V})$

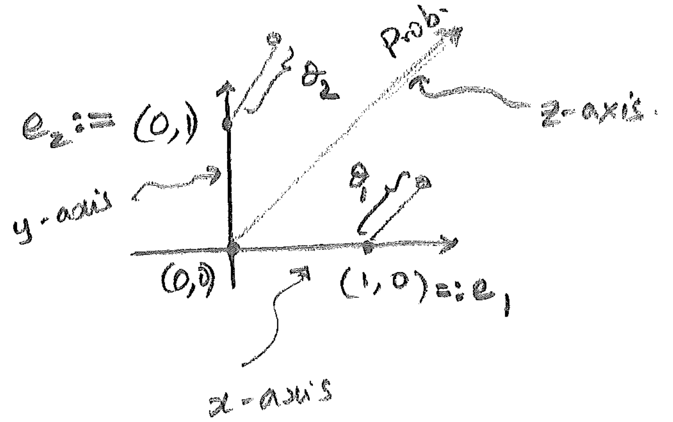
Given a parameter $\theta_1 \in [0,1]$ and $\theta_2 = 1 - \theta_1$, we say $X := (X_1, X_2)$ is Bernoulli(θ_1, θ_2) $R\vec{V}$ if its realisation is:

$$x := (x_1, x_2) \in \{ \overset{e_1}{(1,0)}, \overset{e_2}{(0,1)} \}$$

and its probability mass function is:

$$f(x;\theta_1,\theta_2) := P(X=x) = \theta_1 \mathbb{1}_{\{e_1\}}(x) + \theta_2 \mathbb{1}_{\{e_2\}}(x)$$

$$= \begin{cases} \theta_1, & \text{if } x = e_1 := (1,0) \\ \theta_2 = 1-\theta_1, & \text{if } x = e_2 := (0,1) \\ 0, & \text{otherwise.} \end{cases}$$



—x—

Now let us add independent and identical realisations of Bernoulli (θ_1, θ_2) $R\vec{V}$s to get the Binomial $R\vec{V}$.

<u>Model</u> <u>Binomial (n, θ_1, θ_2) $R\vec{V}$</u>

The Binomial(n,θ_1,θ_2) $R\vec{V}$ $Y := (Y_1, Y_2)$ is the sum of n IID Bernoulli(θ_1,θ_2) $R\vec{V}$s.

Let,

$$X_1 := (X_{11}, X_{12}), \; X_2 := (X_{21}, X_{22}), \; \dots, \; X_n := (X_{n1}, X_{n2})$$

and

$$Y := (Y_1, Y_2) = X_1 + X_2 + \dots + X_n = (X_{11}, X_{12}) + (X_{21}, X_{22}) + (X_{n1}, X_{n2})$$

$$= \sum_{i=1}^{n} (X_{i1}, X_{i2})$$

$$= \left(\sum_{i=1}^{n} X_{i1}, \sum_{i=1}^{n} X_{i2} \right)$$

random position of the ball in the Quincunx after n levels of dropping ... see GUI for reinforcement ...

Read the handout on Permutations, combinations, etc to understand why the PMF of Binomial (n, θ_1, θ_2) $\vec{RV}$ $Y := (Y_1, Y_2)$ with $Y_2 = n - Y_1$ and $\theta_2 = 1 - \theta_1$ is:

$$f((y_1,y_2); n, \theta_1, \theta_2) := P((y_1,y_2); \theta_1, \theta_2) = \binom{n}{y_1} \theta_1^{y_1} (1-\theta_1)^{n-y_1}$$

Recall $\binom{n}{y_1} = \frac{n!}{y_1!(n-y_1)!}$.

Exercise.

Play with the Quincunx (physical Model in Biomaths Pod 6th Floor Erskine and GUI linked from the hand out paper) and realize the meaning of all the symbols in the previous model.

—x—

Model de Moivre $(\theta_1, \theta_2, \dots, \theta_k)$ $\vec{RV}$.

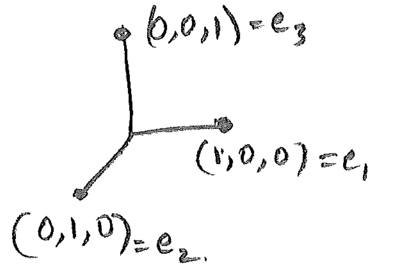
Let the k-dimensional random vector $X := (X_1, X_2, \dots, X_k)$ taking values $x := (x_1, x_2, \dots, x_k) \in \{e_1, e_2, \dots, e_k\}$, where e_i's are the ortho-normal basis vectors, have the PMF:

$$f(x; \theta_1, \theta_2, \dots, \theta_k) := P(X = x) = \sum_{i=1}^{k} \theta_i \mathbb{1}_{\{e_i\}}(x)$$

$$= \begin{cases} \theta_1 & \text{if } x = e_1 := (1, 0, \dots, 0) \in \mathbb{R}^k \\ \vdots & \\ \theta_k & \text{if } x = e_k := (0, 0, \dots, 1) \in \mathbb{R}^k \\ 0 & \text{otherwise.} \end{cases}$$

Note: The parameter space $\Theta = \{(\theta_1, \dots, \theta_k) : \theta_1, \dots, \theta_k \geq 0, \sum_{i=1}^{k} \theta_i = 1\}$

We can visualize realisations of deMoivre$(\theta_1, \theta_2, \theta_3)$ $R\vec{V}$ in 3D.



Exercise: Play with the septcunx (Physical model in BioMaths Pod & GUI) to understand the construction of de Moivre$(\theta_1, \theta_2, \theta_3)$ with $\theta_1 = \theta_2 = \theta_3 = \frac{1}{3}$ as well as the following Model obtained by the sum of IID de Moivre$(\theta_1, \theta_2, \theta_3)$ $R\vec{V}$s. Read the handed out paper
$\longrightarrow$ "Extending Galton's Binomial Quincunx to the Trinomial septcunx".

Model Multinomial$(n, \theta_1, \theta_2, \cdots, \theta_k)$ $R\vec{V}$

$Y := (Y_1, Y_2, \cdots, Y_k)$ is the sum of n IID de Moivre $R\vec{V}$s: $X_i := (X_{i1}, X_{i2}, \cdots, X_{ik})$, $i = 1, 2, \cdots, n$.

$$Y = \sum_{i=1}^{n} X_i$$

Any realisation $y := (y_1, y_2, \cdots, y_k)$ of our $R\vec{V}$ $Y := (Y_1, Y_2, \cdots, Y_k)$ is in the sample space:

$$\mathbb{Y} = \left\{ y := (y_1, \cdots, y_k) : \sum_{i=1}^{k} y_i = n \right\}$$

and its PMF is:

$$P(Y = y) =: f(y; n, \theta_1, \cdots, \theta_k) = \binom{n}{y_1 \cdots y_k} \theta_1^{y_1} \cdots \theta_k^{y_k}$$

where, $\binom{n}{y_1 \cdots y_k} = \dfrac{n!}{y_1! \, y_2! \cdots y_k!}$.

Simulation.

How do we simulate from the Multinomial $R\vec{V}$ Y?

We can simply sum up (do vector addition) samples drawn from de Moivre $R\vec{V}$ $X_1, \ldots, X_n$.

So, we need to be able to draw samples or simulate from de Moivre $(\theta_1, \theta_2, \cdots, \theta_k)$ $R\vec{V}$.

The Algorithm to simulate from de Moivre $(\theta_1, \cdots, \theta_k)$ $R\vec{V}$ is similar to that for Bernoulli (θ) RV.

Algorithm : (Simulate from de Moivre $(1/k, 1/k, \cdots, 1/k)$ RV

input: $u \sim$ Uniform $(0,1)$ PRNG.

k , parameter.

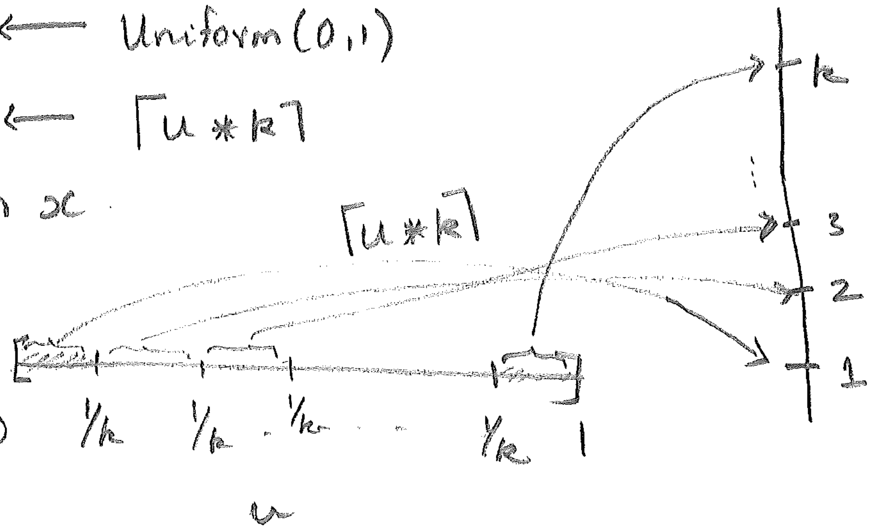
output: $x \sim$ de Moivre $(1/k, \cdots, 1/k)$ RV

$u \longleftarrow$ Uniform $(0,1)$

$x \longleftarrow \lceil u * k \rceil$

return x.

Picture.



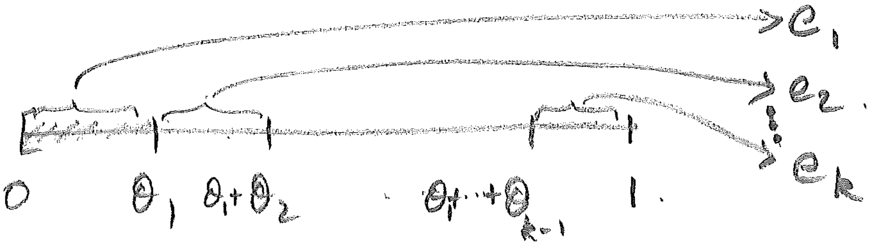
The inverse DF $F^{[-1]}(u) : [0,1] \rightarrow \{e_1, \cdots, e_k\}$ for de Moivre $(\theta_1, \theta_2, \cdots, \theta_k)$ $R\vec{V}$ is :

$$F^{[-1]}(u; \theta_1, \cdots, \theta_k) = \begin{cases} e_1 & \text{if } 0 \leq u < \theta_1 \\ e_2 & \text{if } \theta_1 \leq u < \theta_1 + \theta_2 \\ e_3 & \text{if } \theta_1 + \theta_2 \leq u < \theta_1 + \theta_2 + \theta_3 \\ \vdots & \\ e_k & \text{if } \theta_1 + \theta_2 + \cdots + \theta_{k-1} \leq u < 1 \end{cases}$$

Thus, all we need to simulate from de Moivre$(\theta_1, \ldots, \theta_k)$ $R\vec{V}$ is a for loop.

<u>exercise</u> (*) implement a for loop that will take the former expression for $F^{[-1]}(u)$ and produce a sample from de Moivre$(\theta_1, \ldots, \theta_k)$ $R\vec{V}$.

Picture you need to have is:



<u>Hint</u>: To simulate from de Moivre$(\frac{1}{k}, \ldots, \frac{1}{k})$ $R\vec{V}$ X you can modify the earlier Algorithm to simulate from de Moivre$(\frac{1}{k}, \ldots, \frac{1}{k})$ RV X as follows:

$u \longleftarrow$ Uniform$(0,1)$

$x \longleftarrow e_{\lceil u * k \rceil}$

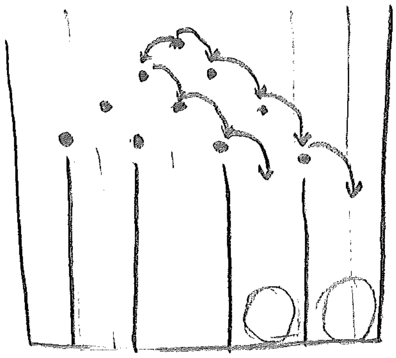
Finally, to simulate from Multinomial$(n, \theta_1, \ldots, \theta_k)$ $R\vec{V}$ Y we can do the following (body of the Algorithm):

$y = (0, 0, \ldots, 0)$
for i : 1 to n.
  $u \leftarrow$ Uniform$(0,1)$
  $x \leftarrow F^{[-1]}(u; \theta_1, \ldots, \theta_k)$ ⟵ This is <u>exercise (*)</u>
  $y \leftarrow y + x$
end for
return y

<u>exercise</u>: complete the Algorithm to simulate from Multinomial$(n, \theta_1, \ldots, \theta_k)$ $R\vec{V}$ sketched above in detail. (including the additional for loop in exercise (*)).

Inference with the Quincunx:

Qn: Suppose you observe some samples from the balls dropped into the Quincunx.



$y_1 = 4$, $y_2 = 3$

Exercise:

What is the MLE of θ_1 ?

Let $\Theta_1 = [0,1]$, you know $\theta_2 = 1 - \theta_1$.

Answer 1:

Hint:

$$Y_1, Y_2 \overset{IID}{\sim} \text{Binomial}(n=4, \theta_1, \theta_2 = 1-\theta_1).$$

$$\text{MLE of } \theta_1 = \underset{\theta_1 \in \Theta = [0,1]}{\text{argmax}} \; f(\underbrace{y_1}_{4}; 4, \theta_1) * f(\underbrace{y_2}_{3}; 4, \theta_1)$$

⋮

Answer 2:

Alternatively, you can use the fact that the Binomial is the sum of Bernoulli's, and write the experiment appropriately so that you only need to know the total # of right turns...

Hint:

$$X_1, \cdots, X_8 \overset{IID}{\sim} \text{Bernoulli}(\theta_1)$$

$$\text{MLE of } \theta_1 = \underset{\theta_1 \in [0,1]}{\text{argmax}} \; \underbrace{\prod_{i=1}^{8} f(x_i; \theta_1)}_{\theta_1^{\sum_{i=1}^{8} x_i} (1-\theta_1)^{8 - \sum_{i=1}^{8} x_i}}$$

...