# Random Variable (RV).

Some real-world examples of RVs

Discrete RVs

Ex 1:

The number on the first ball that pops out of the NZ Lotto machine.

Ex 2:

whether the number on the first ball in Ex 1. is odd?

Ex 3:

Roll two dice and record the number on the top faces.

Continuous RV

Ex 4:

The position of a pollen grain released at the headwaters of Waimakiriri in micrometers above sea-level.

Both Discrete & Continuous RVs.

Ex 5:

The number and length of hairs on my head right now. (actually, random vectors with randomly many components)

Ex 6:

The nature and extent of how every one here is inter-related in space-time. (This is my research area mathematical & statistical genetics.)

(actually, random structures)

In all examples above, we assign real numbers to our observations. Let us define RV formally next

## DFn Random Variable

Let $(\Omega, \mathcal{F}, P)$ be some probability triple. Then, a random variable (RV), say $X$, is a function from the sample space $\Omega$ to the set of real numbers $\mathbb{R}$

(contd...)

(Dfn of RV contd...). 

$$X : \Omega \longrightarrow \mathbb{R}$$

such that for every $x \in \mathbb{R}$, the inverse image of the half-open interval $(-\infty, x]$ is an element of the collection of events $\mathcal{F}$, i.e.:

$$\text{for every } x \in \mathbb{R}, \quad X^{[-1]}((-\infty, x]) := \{\omega : X(\omega) \le x\} \in \mathcal{F}.$$

Finally, we assign probability to the RV $X$ as follows:

$$P(X \le x) = P(X^{[-1]}((-\infty, x])) := P(\{\omega : X(\omega) \le x\}).$$

— x —

This assignment of probability to RVs natually leads to the next definition.

Dfn The Distribution Function (DF) or Cumulative Distribution Function (CDF) of any RV $X$ over a probability triple $(\Omega, \mathcal{F}, P)$, denoted by $F$ is:

$$F(x) := P(X \le x) = P(\{\omega : X(\omega) \le x\}), \text{ for any } x \in \mathbb{R}.$$

Math. Note: Thus, $F(x)$ or simply $F$ is a non-decreasing, right continuous, $[0,1]$-valued function over $\mathbb{R}$.

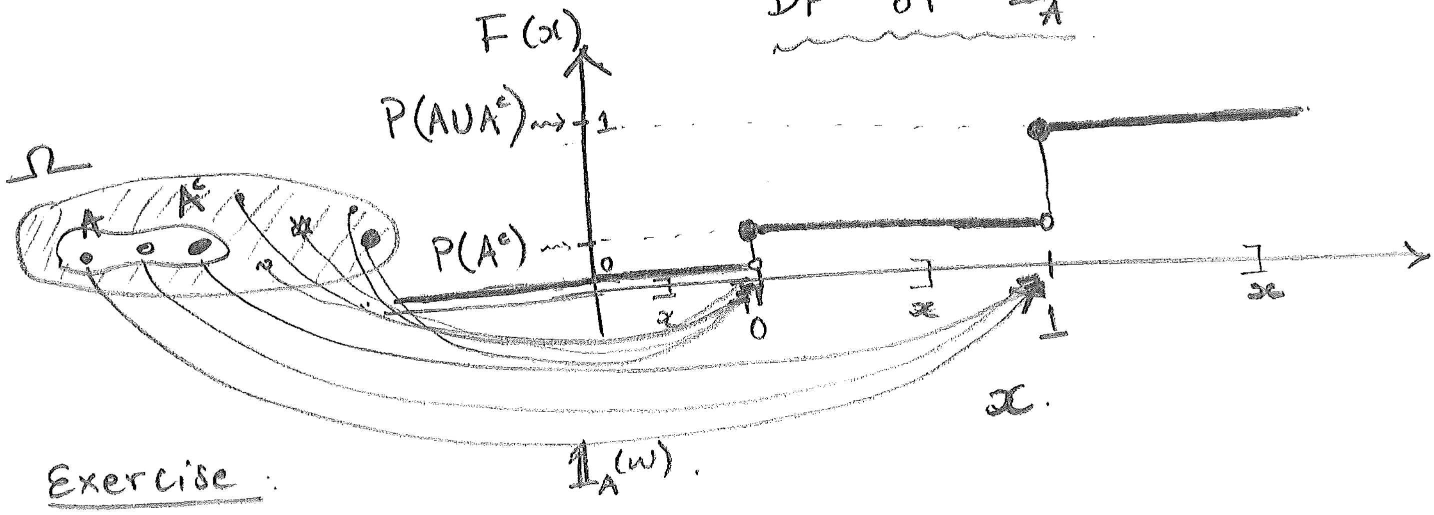
— x —

Now, let us look at a special RV that is a basic building-block in Probability theory, Measure Theory and Statistical theory. It is the indicator function or the Bernoulli RV. 

Dfn The indicator function of an event $A \in \mathcal{F}$, denoted $\mathbb{1}_A$ is defined as follows:

$$\mathbb{1}_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

DF of $\mathbb{1}_A$




Exercise:

Let us convince ourselves that $\mathbb{1}_A$ is really a RV. Recall that for $\mathbb{1}_A$ to be a RV, we need to verify that for any real number $x \in \mathbb{R}$, the inverse image $\mathbb{1}_A^{[-1]}((-\infty, x])$ is an event, i.e.:

$$\mathbb{1}_A^{[-1]}((-\infty, x]) := \{\omega : \mathbb{1}_A(\omega) \leq x\} \in \mathcal{F}.$$

All that we can assume about the collection of events $\mathcal{F}$ is that it is a sigma Algebra and that it contains $A$.

$$\mathbb{1}_A^{[-1]}((-\infty, x]) := \{\omega : \mathbb{1}_A(\omega) \leq x\} = \begin{cases} \emptyset & , \text{if } x < 0 \\ A^c & , \text{if } 0 \leq x < 1 \\ A \cup A^c = \Omega, & \text{if } x \geq 1. \end{cases}$$

contd...

Thus, $\mathbb{1}_A^{[-1]}((-\infty, x])$ is one of the following three sets that have to belong to $\mathcal{F}$;

$$\emptyset$$
$$A^c$$
$$A \cup A^c = \Omega$$

depending on the value taken by $x$ relative to the interval $[0,1]$. We have proved that $\mathbb{1}_A$ is indeed a RV.

EX: 'Will it rain tomorrow in the S. Alps?' can be formulated as the RV given by the indicator function of the event 'rain drops fall on S. Alps tomorrow'. Can you imagine what the little omegas '$\omega$'s in the sample space $\Omega$ can be?

Next we introduce a $\theta$-parametrized family of $\mathbb{1}_A$ called the Bernoulli RV. First, we get introduced more formally to a kind of RV called discrete RV (see Exs. in page 1).

Dfn: When a RV takes at most countably many values in $\mathbb{R}$ it is said to be a discrete RV.

Dfn: Let $X$ be a discrete RV over a probability triple $(\Omega, \mathcal{F}, P)$. The probability Mass Function (PMF) $f$ of $X$ is:

$$f(x) := P(X=x) = P(\{\omega : X(\omega) = x\}).$$

The indicator function $\mathbb{1}_A$ of the event that 'A occurs' for the $\theta$-specific probability triple $(\Omega, \mathcal{F}, P_\theta)$, with $A \in \mathcal{F}$, is the Bernoulli$(\theta)$ RV. The parameter $\theta$ denotes the probability that 'A occurs', i.e. $P(A) = \theta$.

Model [Bernoulli$(\theta)$]

Given a parameter $\theta \in [0,1]$, the probability mass function (PMF) for the Bernoulli$(\theta)$ RV $X$ is:

$$f(x;\theta) = \theta^x (1-\theta)^{1-x} \mathbb{1}_{\{0,1\}}(x) = \begin{cases} \theta & \text{if } x = 1 \\ 1-\theta & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

and its DF is:

$$F(x;\theta) = \begin{cases} 1 & \text{if } 1 \le x \\ 1-\theta & \text{if } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Note:

We emphasise the dependence of the probabilities of events on the parameter $\theta$ by specifying it following the semicolon in the argument for PDF $f$ and DF $F$. Sometimes, we also subscript the probabilities by parameter. e.g. $P_\theta(X=1) = \theta$ and $P_\theta(X=0) = 1-\theta$

# An Elementary Continuous Random Variable.

When a random variable takes values in the continuum we call it a continuous RV.

Ex: Vertical position (above sea-level) in micrometers since the original release of a pollen grain on Waimakiriri

Ex: Diameter of a randomly sampled Coarse Venus shell by New Brighton Pier.

Ex. Volume of water in $m^3$ that fell on S. Alps last year.

Continuous RVs require a sophisticated way of talking about probability — now we need Integrals & Differentials from calculus.

Dfn: Probability Density Function (PDF).

A RV $X$ with D.F. $F$ is called continuous if there exists a piece-wise continuous function $f$, called the probability density function (PDF) of $X$, such that for any $a, b \in \mathbb{R}$ with $a < b$,

$$P(a < X \leq b) = F(b) - F(a) = \int_a^b f(x)\, dx.$$

Properties of a continuous RV $X$ with PDF $f$ and DF $F$:

1) For any $x \in \mathbb{R}$, $P(X = x) = 0$

ex: $P($ Venus shell $= 1.0$ cm$) = 0$

2) Due to 1):
for any $a, b \in \mathbb{R}$ with $a \leq b$,

$$P(a < X < b) = P(a < X \leq b) = P(a \leq X \leq b) = P(a \leq X < b)$$

3) By the Fundamental Theorem of Calculus (except possibly at finitely many points where the continuous pieces come together in the piece-wise continuous $f$):

$$f(x) = \frac{d}{dx} F(x) = dF(x)$$

abbreviated

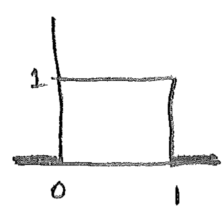
4) $f$ must satisfy:

$$\int_{-\infty}^{\infty} f(x)\, dx = P(-\infty < X < \infty) = 1.$$

<u>Model</u> The Fundamental Model or Uniform(0,1) RV.

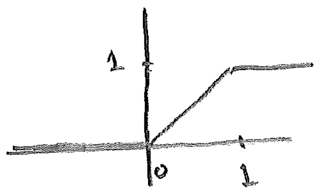
The PDF of Uniform(0,1) RV $X$ is:

$$f(x) = \mathbb{1}_{[0,1]}(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$




and its DF is:

$$F(x) = \int_{-\infty}^{x} f(y)\, dy = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x > 1 \end{cases}$$




<u>Note</u>: $F$ is the identity map in $[0,1]$.

# Expectations

Dfn: Expectation of a function $g$ of a RV $X$ with DF $F$ is

$$E(g(X)) := \int g(x)\, dF(x) = \begin{cases} \sum_x x \cdot f(x) & \text{if } X \text{ is a discrete RV} \\ \int_{-\infty}^{\infty} g(x) f(x)\, dx & \text{if } X \text{ is a continuous RV.} \end{cases}$$

provided the sum or integral is well-defied. We say the expectation exists if:

$$\int_{-\infty}^{\infty} |g(x)|\, dF(x) < \infty$$

↑ absolute value

(i) Three Special Expetations

Parametric Case: $E_\theta(g(X)) := \int g(x)\, dF(x;\theta)$

$\boxed{g(x) = x}$

Expectation of X (Population Mean, first moment, Expected value).

$$E(X) = \int x\, dF(x)$$

(ii)

$\boxed{g(x) = (x - E(X))^2}$

Variance of X

$$V(X) := E\left((X - E(X))^2\right) = \int (x - E(X))^2\, dF(x)$$

std deviation $= \sqrt{V(X)}$.

(iii) $\boxed{g(x) = x^k}$

k-th moment of a RV is

$$E(X^k) = \int x^k \, dF(x)$$

The k-th moment of X is said to exist if

$$E(|X|^k) < \infty.$$

Properties of Expectations.

1. If the k-th moment exists and if $j < k$ then the j-th moment exists.

2. If $X_1, X_2, \ldots, X_n$ are RVs and $a_1, a_2, \ldots, a_n$ are constants, then

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$$

3. Let $X_1, X_2, \ldots, X_n$ be independent RVs, Then

$$E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$$

4. $V(X) = E(X^2) - (E(X))^2$ Exercise [prove by Completing The square].

5. If a and b are constants, then

$$V(aX+b) = a^2 V(X)$$

6. If $X_1, X_2, \ldots, X_n$ are independent and $a_1, a_2, \ldots, a_n$ are constants, then:

$$V\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 V(X_i).$$