Probability

Origins

The subject of probability can be traced back to 17th century. It arose out of the study of gambling and games of chance.

Language for Probability:

An **experiment** is an activity ~~or procedure~~ that produces distinct or well defined **outcomes**. The set of such outcomes is called the **sample space** of the experiment (usually denoted by Ω).

EX 1: 'Roll a dice' Experiment with faces painted 'Red', 'Green', 'yellow', 'Orange', 'blue', 'black'.

Sample space or Ω = {Red, Green, yellow, Orange, blue, black}.

EX 2 'Flip a coin' Experiment

$$\Omega = \{H, T\}.$$

EX 3 'Draw a fruit from a Fruit bowl (well mixed) with 2 oranges, 3 apples and 1 lemon' Experiment.

$$\Omega \overset{?}{=} \{\text{Orange}, \text{apple}, \text{lemon}\}.$$

An **Event** is a subset of the sample space. Ex The event {orange, lemon} $\subseteq \Omega$ in Ex 3

Abstractly, **Probability** is a **function** P: set of events $\to [0,1]$ that assigns numbers to events, which satisfies the following four **axioms**:

(i) For any event A, $0 \le P(A) \le 1$.

(ii) If Ω is the sample space, $P(\Omega) = 1$

(iii) If A and B are disjoint, i.e., $A \cap B = \phi$, then

$$P(A \cup B) = P(A) + P(B)$$

(iv) If $A_1, A_2, \dots$ is an infinite sequence of pair-wise disjoint events (i.e., $A_i \cap A_j = \emptyset$ when $i \neq j$), Then

$$\underbrace{P\left(\bigcup_{i=1}^{\infty} A_i\right)}_{\parallel \atop A_1 \cup A_2 \cup A_3 \cup \cdots} = \underbrace{\sum_{i=1}^{\infty} P(A_i)}_{\parallel \atop P(A_1)+P(A_2)+P(A_3)+\cdots}$$

These axioms or assumptions are motivated by the frequency interpretation of probability, which states that if we repeat an experiment a large number of times then the fraction of times the event A occurs will be close to P(A). More precisely,

Let $N(A,n)$ = the number of times A occurs in the first n trials,

Then

$$P(A) = \underbrace{\lim_{n\to\infty} \frac{N(A,n)}{n}}$$

$$\frac{N(A,1)}{1}, \frac{N(A,2)}{2}, \frac{N(A,3)}{3}, \ldots$$

where is this fraction going

Let us make sure (i), (ii) & (iii) make sense

clearly

$$0 \le \frac{N(A,n)}{n} \le 1$$

something happens

$$P(\Omega) = \frac{N(\Omega,n)}{n} = \frac{n}{n} = 1$$

If $A \cap B = \emptyset$,

Then

$$N(A\cup B,n) = N(A,n) + N(B,n)$$

Since $A \cup B$ occurs if either A or B occurs but it is impossible for both to happen. (This extends to finitely many disjoint events).

(iv) is a more controversial axiom. here we will assume it as part of our axiomatic dfn of probability, since without it the maths is much harder.

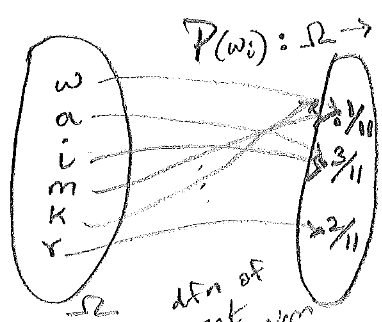
{Do Ex 1, Ex 2 & Ex 3 in order}

Ex 3 Suppose we pick a letter at random from well-stirred waimakariri Urn of letters.

What is The sample space Ω and what probabilities should be assigned to the outcomes?

$$\Omega = \{\overset{\omega_1}{w}, \overset{\omega_2}{a}, \overset{\omega_3}{i}, \overset{\omega_4}{m}, \overset{\omega_5}{k}, \overset{\omega_6}{r}\}.$$

$P(\omega_i) : \Omega \to [0,1]$



$P(\{w\}) = 1/11$, $P(\{a\}) = \frac{3}{11}$, $P(\{i\}) = \frac{3}{11}$,
$P(\{m\}) = 1/11$, $P(\{k\}) = \frac{1}{11}$, $P(\{r\}) = \frac{2}{11}$

dfn of set union; these are pairwise disjoint events; by axiom (iii) extended to 3 events.

$$P(\{w,a,r\}) \overset{?}{=} P(\{w\} \cup \{a\} \cup \{r\}) \overset{?}{=} P(\{w\}) + P(\{a\}) + P(\{r\})$$
$$= \frac{1}{11} + \frac{3}{11} + \frac{2}{11} = \frac{6}{11}$$

$\boxed{\text{So}}$:- $P(A) : \sigma(\Omega) \xrightarrow{?} [0,1]$, $\sigma(\Omega)$ is called σ-algebra or σ-field (more later...)

Ex 2 NZ Lotto : $\Omega = \{1, 2, \ldots, 40\}$ and $P(\omega) = \frac{1}{40}$ for each $\omega \in \Omega$

i.e., $P(1) = P(2) = \cdots = P(40) = \frac{1}{40}$

(i) What is Probability of the first ball drawn being even?

$$P(\{2,4,6,\ldots,36,38,40\})$$

dfn of set union

$$= P(\{2\} \cup \{4\} \cup \{6\} \cup \cdots \cup \{36\} \cup \{38\} \cup \{40\})$$

due to extended axiom (iii)

$$\overset{?}{=} P(\{2\}) + P(\{4\}) + P(\{6\}) + \cdots + P(\{36\}) + P(\{38\}) + P(\{40\})$$

$$= \sum_{i \in \{2,4,6,\ldots,36,38,40\}} P(i) = 20 * \frac{1}{40} = \frac{1}{2}$$

and

$$P(\text{odd ball}) = P(\{1,3,\ldots,37,39\}) = P(\{1\}) \cup P(\{3\}) \cup \cdots P(\{39\}) = \frac{20}{40}$$

Ex 1 Toss a fair coin: $\Omega = \{H, T\}$ $P(H) = P(T) = 1/2$

check that all axioms are satisfied by our $P : \{H,T\} \to [0,1]$

$$P(\Omega) = P(\{H,T\}) = P(\{H\}) + P(\{T\}) = \frac{1}{2} + \frac{1}{2} = 1$$

(iii)

(i) $0 \le P(H) = P(T) = \frac{1}{2} \le 1$

$0 \le P(\Omega) = 1 \le 1$

(ii)

(iv) o.k too

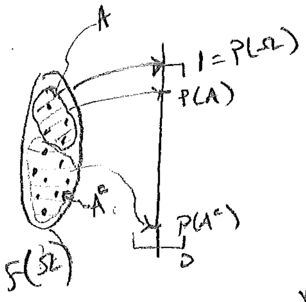
Having introduced a number of dfns, we now derive some basic logical consequences (i.e., properties) of probabilities (axiomatically defined).

<u>Property 1</u>: $P(A) = 1 - P(A^c)$, $A^c = \Omega \setminus A$

<u>Proof</u>: Let $A_1 = A$ and $A_2 = A^c$. [just renaming for convenience]

Then, $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = \Omega$

Proof by picture



So, (iii) implies $P(A) + P(A^c) = P(\Omega) = 1$ ⊛ by (ii)

Recall

$A_1 \cap A_2 = \emptyset \Rightarrow P(A_1 \cup A_2) = P(A_1) + P(A_2)$

recall! $P(\Omega) = 1$.

subtracting $P(A^c)$ from either side of $P(A) + P(A^c) = 1$.

$P(A) \cancel{+ P(A^c)} \cancel{- P(A^c)} = 1 - P(A^c)$

Finally, a rearrangement of terms in ⊛ yields: $\boxed{P(A) = 1 - P(A^c)}$ Q.E.D.

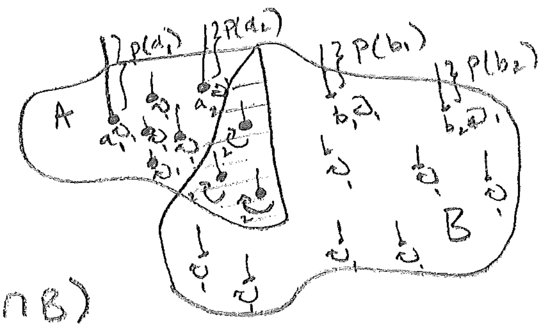
<u>Ex</u>:- $\Omega = \{H, T\}$ $P(H) = 1 - P(\Omega \setminus H) = 1 - P(T)$

<u>Property 2</u>: For any two events A, B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

<u>Proof by Picture</u>:

$P(A \cup B) = P(A)$
$+ P(B)$
$- P(A \cap B)$



correcting the double-counting/summing in $A \cap B$. ▭