0.1 Permutations, Factorials and Combinations

Definition 1 (Permutations and Factorials) *A* **permutation** *of n objects is an arrangement of n distinct objects in a row. For example, there are 2 permutations of the two objects $\{1,2\}$:*

$$12, \qquad 21 \ ,$$

and 6 permutations of the three objects $\{a,b,c\}$:

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba \ .$$

Let the number of ways to choose k objects out of n and to arrange them in a row be denoted by $p_{n,k}$. For example, we can choose two ($k=2$) objects out of three ($n=3$) objects, $\{a,b,c\}$, and arrange them in a row in six ways ($p_{3,2}$):

$$ab, \quad ac, \quad ba, \quad bc, \quad ca, \quad cb \ .$$

Given n objects, there are n ways to choose the left-most object, and once this choice has been made there are $n-1$ ways to select a different object to place next to the left-most one. Thus, there are $n(n-1)$ possible choices for the first two positions. Similarly, when $n>2$, there are $n-2$ choices for the third object that is distinct from the first two. Thus, there are $n(n-1)(n-2)$ possible ways to choose three distinct objects from a set of n objects and arrange them in a row. In general,

$$p_{n,k} = n(n-1)(n-2)\dots(n-k+1)$$

and the total number of permutations called 'n **factorial***' and denoted by $n!$ is*

$$n! := p_{n,n} = n(n-1)(n-2)\dots(n-n+1) = n(n-1)(n-2)\dots(3)\ (2)\ (1) =: \prod_{i=1}^{n} i \ .$$

Some factorials to bear in mind

$$0! := 1 \quad 1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120 \quad 10! = 3,628,800 \ .$$

When n is large we can get a good idea of $n!$ without laboriously carrying out the $n-1$ multiplications via Stirling's approximation (*Methodus Differentialis (1730), p. 137*) :

$$n! \approxeq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \ .$$

Definition 2 (Combinations) *The combinations of n objects taken k at a time are the possible choices of k different elements from a collection of n objects, disregarding order. They are called the k-combinations of the collection. The combinations of the three objects $\{a,b,c\}$ taken two at a time, called the 2-combinations of $\{a,b,c\}$, are*

$$ab, \quad ac, \quad bc \ ,$$

and the combinations of the five objects $\{1,2,3,4,5\}$ taken three at a time, called the 3-combinations of $\{1,2,3,4,5\}$ are

$$123, \quad 124, \quad 125, \quad 134, \quad 135, \quad 145, \quad 234, \quad 235, \quad 245, \quad 345 \ .$$

The total number of k*-combination of* n *objects, called a* **binomial coefficient***, denoted* $\binom{n}{k}$ *and read "*n *choose* k*," can be obtained from* $p_{n,k} = n(n-1)(n-2)\dots(n-k+1)$ *and* $k! := p_{k,k}$*. Recall that* $p_{n,k}$ *is the number of ways to choose the first* k *objects from the set of* n *objects and arrange them in a row with regard to order. Since we want to disregard order and each* k*-combination appears exactly* $p_{k,k}$ *or* $k!$ *times among the* $p_{n,k}$ *many permutations, we perform a division:*

$$\binom{n}{k} := \frac{p_{n,k}}{p_{k,k}} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k(k-1)(k-2)\dots 2\ 1} .$$

Binomial coefficients are often called "Pascal's Triangle" and attributed to Blaise Pascal's *Traité du Triangle Arithmétique* from 1653, but they have many "fathers". There are earlier treatises of the binomial coefficients including Szu-yüan Yü-chien ("The Precious Mirror of the Four Elements") by the Chinese mathematician Chu Shih-Chieh in 1303, and in an ancient Hindu classic, *Piṅgala's Chandaḥśāstra*, due to Halāyudha (10-th century AD).