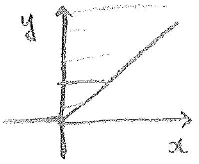
Soln: Draw the support of (X, Y)



$$f_Y(y)=\int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} 0 & \text{if } y \le 0 \\ >0 & \text{if } y>0 \end{cases}$$

So, for $y>0$:

$$f_Y(y)=\int_{x=0}^{y}\frac{6}{10^6}e^{-0.001x-0.002y}\,dx$$

$$=\frac{6}{10^6}e^{-0.002y}\underbrace{\int_{x=0}^{y}e^{-0.001x}\,dx}$$

$$= \left[\frac{1}{-0.001}e^{-0.001x}\right]_{x=0}^{y}$$

$$= \frac{e^{-0.001y}}{-0.001} - \frac{1}{-0.001}$$

$$= \frac{1}{0.001} - \frac{e^{-0.001y}}{0.001}$$

$$= \frac{6}{10^6}e^{-0.002y}\left(\frac{1-e^{-0.001y}}{0.001}\right)$$

So, $f_Y(y) = \frac{6}{10^3}e^{-0.002y}\left(1-e^{-0.001y}\right)$

From this we can get $P(Y>2000)=\int_{2000}^{\infty}f_Y(y)\,dy$

$$=\frac{6}{10^3}\int_{2000}^{\infty}e^{-0.002y}\left(1-e^{-0.001y}\right)dy=\frac{6}{10^3}\left[\int_{2000}^{\infty}e^{-0.002y}\,dy-\int_{2000}^{\infty}e^{-0.003y}\,dy\right]$$

$$=\frac{6}{10^3}\left(\left[\frac{e^{-0.002y}}{-0.002}\right]_{y=2000}^{\infty}-\left[\frac{e^{-0.003y}}{-0.003}\right]_{y=2000}^{\infty}\right)=\frac{6}{10^3}\left(\frac{e^{-4}}{0.002}-\frac{e^{-6}}{0.003}\right)=0.05$$

§14.2 Expectations of functions of bivariate $\vec{RV}$s. (4)

Dfn. The expectation of a function $g(X,Y)$ of a $\vec{RV}$ (X,Y) is:

$$E(g(X,Y)) = \begin{cases} \sum_{(x,y)\in S_{X,Y}} g(x,y) f_{X,Y}(x,y) & \text{if } (X,Y) \text{ is discrete } \vec{RV} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy & \text{if } (X,Y) \text{ is continuous } \vec{RV} \end{cases}$$

Some typical expectations are:

1. Joint Moments $\quad E(X^r Y^s) \quad ; g(x,y) = x^r y^s$

2. Need new notion of variance of two RVs

If $E(X^2) < \infty$, $E(Y^2) < \infty$ then $E(|XY|) < \infty$ and $E(|(X-E(X))(Y-E(Y))|) < \infty$, allowing us to define <u>covariance</u> of X and Y as:

$$\mathrm{Cov}(X,Y) := E((X-E(X))(Y-E(Y)))$$

✡. $$\boxed{\mathrm{Cov}(X,Y) \overset{\vdots}{=} E(XY) - E(X)E(Y)}$$ expanding rectangle & taking expectations

—x—

Two RVs X and Y are said to be <u>independent</u> if and only if for every (x,y)

$$F_{X,Y}(x,y) = F_X(x)\,F_Y(y) \quad \text{or} \quad f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$$

Ex Consider $R\vec{V}$ $(X,Y) \sim Uniform([0,1]^2)$ of Ex 14.5

Qn. Are X and Y independent RVs ?

To check if X and Y are independent let's check if the JPDF $f_{X,Y}(x,y)$ is the same as the product of the marginal PDFs $f_X(x)\, f_Y(y)$ for each $(x,y) \in \mathbb{R}^2$.

Recall

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,1]^2 \\ 0 & \text{o.w.} \end{cases}$$

$$f_X(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{o.w.} \end{cases} \quad , \quad f_Y(y) = \begin{cases} 1 & \text{if } y \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$

Therefore

$$\begin{cases} 1 = f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = 1 * 1 = 1 & \text{if } (x,y) \in [0,1]^2 \\ 0 = \quad '' \quad = \quad '' \quad = 0 * 0 = 0 & \text{if } (x,y) \notin [0,1]^2 \end{cases}$$

Since, $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$ for every $(x,y) \in \mathbb{R}^2$

X and Y are independent

Ex. Are X and Y independent in the server times $R\vec{V}$ (X,Y) of Ex 14.7 ?

Ans. No. Can check $f_{X,Y}(x,y) \neq f_X(x)\, f_Y(y)$ for every $(x,y) \in \mathbb{R}^2$

[Restriction that $Y > X$, makes it dependent].

[$f_{X,Y}(x,y) = 0$ if $x > y$, but $f_X(x)\, f_Y(y) > 0$ if $x > y$.

2 **Some useful Properties of Expectations.**

(in addition to those from EMTH119 p.59)

If X and Y are independent RVs

- $E(XY) = E(X)E(Y)$
- $E(g(X)h(Y)) = E(g(X))\,E(h(Y))$
- $E(aX+bY+c) = aE(X) + bE(Y) + c$
- $V(aX+bY+c) = a^2V(X) + b^2V(Y)$

If X and Y are any two RVs (indep or not)

- $E(aX+bY+c) = aE(X) + bE(Y) + c$
- $V(aX+bY+c) = a^2V(X) + b^2V(Y) + 2ab\,Cov(X,Y)$

1

§ 14.3 Multivariate $R\vec{V}_s$.

The bivariate Dfns naturally extend to m-variate case for m-tuple $(X_1, \dots, X_m)$ [see notes].

JDF
$$F_{X_1,\dots,X_m}(x_1,\dots,x_m) = P(X_1 \le x_1, X_2 \le x_2, \dots, X_m \le x_m)$$

for each $(x_1, \dots, x_m) \in \mathbb{R}^m$.

The 4 properties also hold: (1) $F_{X_1,\dots,X_m} \in [0,1]$.

(2) JDF is non-decreasing function of $(x_1, \dots, x_m)$

(3) $F_{X_1,\dots,X_m}(x_1,\dots,x_m) \to 1 \,/ \to 0$ as $(x_1, \dots, x_m) \to (+\infty, \dots, +\infty) \,/ \to (-\infty, \dots$

Dfn 14.14 JPMF of discrete $R\vec{V}$ $(X_1, \dots, X_m)$ is

$$f_{X_1,\dots,X_m}(x_1,\dots,x_m) = P(X_1 = x_1, X_2 = x_2, \dots, X_m = x_m)$$

Dfn 14.15 JPDF of contin. $R\vec{V}$ $(X_1, \dots, X_m)$ is

$$f_{X_1,\dots,X_m}(x_1,\dots,x_m) = \frac{\partial^m}{\partial x_1 \partial x_2 \cdots \partial x_m} F_{X_1,\dots,X_m}(x_1,\dots,x_m)$$

Again:

$$F_{X_1,\dots,X_m}(x_1,\dots,x_m) = \int_{-\infty}^{x_m} \cdots \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f_{X_1,\dots,X_m}(u_1,u_2,\dots,u_m)\, du_1\, du_2 \cdots du_m$$

$$P(B) = \int\cdots\iint_B f_{X_1,\dots,X_m}(x_1,\dots,x_m)\, dx_1 \cdots dx_m.$$

etc:

Marginal PMF / PDF comes from summing / integrating the JPMF / JPDF as usual.

$$f_{X_1}(x_1) = \int_{x_m=-\infty}^{\infty} \cdots \int_{x_2=-\infty}^{\infty} f_{X_1,\dots,X_m}(x_1,\dots,x_m)\, dx_2 \cdots dx_m$$

14.3.1. m Independent RVs

$X_1, \dots, X_m$ are jointly independent if and only if for every $(x_1, \dots, x_m) \in \mathbb{R}^m$

$$F_{X_1,\dots,X_m}(x_1,\dots,x_m) = F_{X_1}(x_1) \cdots F_{X_2}(x_m)$$

or $$f_{X_1,\dots,X_m}(x_1,\dots,x_m) = f_{X_1}(x_1) \cdots f_{X_m}(x_m)$$

EX 14.16 If X_1 & X_2 are indep. RVs then what is their covariance?

by indep. $E(X_1X_2) = E(X_1)E(X_2)$ so,

$$\text{cov}(X_1,X_2) = E(X_1X_2) - E(X_1)E(X_2)$$
$$= E(X_1)E(X_2) - E(X_1)E(X_2)$$
$$= 0.$$

14.3.2 Linear Combination of Independent Normal RVs is Normal.

IF $X_1, \ldots, X_m$ are jointly independent RVs

with $X_i \sim Normal(\mu_i, \sigma_i^2)$ for each $i \in \{1, \ldots, m\}$

Then $\boxed{Y = c + \sum_{i=1}^{m} a_i X_i}$ for some constants $c, a_1, \ldots, a_m$

is $\boxed{Normal\left(c + \sum_{i=1}^{m} a_i \mu_i \, , \, \sum_{i=1}^{m} a_i^2 \sigma_i^2\right)}$ RV.

Ex 14.18

Let $X \sim Normal(2, 4)$, $Y \sim Normal(-1, 2)$

$Z \sim Normal(0, 1)$ be jointly independent.

1. $E(3X - 2Y + 4Z) = 3E(X) - 2E(Y) + 4E(Z)$

$= 3 * 2 - 2 * (-1) + 4 * 0$

$= 6 + 2 + 0 = 8$

2. $V(2Y - 3Z) = 2^2 V(Y) + (-3)^2 V(Z) = 4 * 2 + 9 * 1$

$= 8 + 9 = 17.$

3. Find the distribution of: $6 - 2Z + X - Y$.

$6 - 2Z + X - Y$ is

Normal RV. with mean $= 6 - 2*0 + 2 - (-1) = 6 - 0 + 2 + 1 = 9$

and variance $= (-2)^2 * 1 + 1^2 * 4 + (-1)^2 * 2$

$= 4 + 4 + 2 = 10$

So, $6 - 2Z + X - Y \sim Normal(9, 10)$.

4. Let $U = 6 - 2Z + X - Y$ and from 3 we know

$U \sim \text{Normal}(9, 10)$.

find $P(6 - 2Z + X - Y > 0) = P(U > 0)$

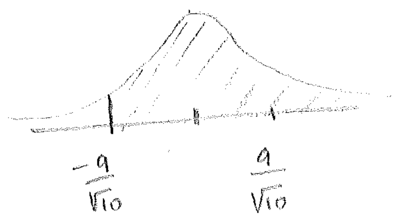
$$= P(U - 9 > 0 - 9)$$

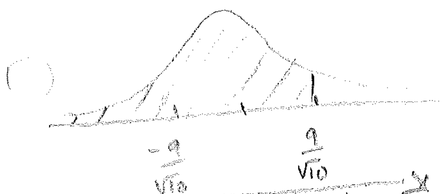
$$= P\left(\frac{U-9}{\sqrt{10}} > \frac{-9}{\sqrt{10}}\right) = P\left(Z > \frac{-9}{\sqrt{10}}\right)$$

$$= P\left(Z < \frac{9}{\sqrt{10}}\right)$$

$$\approx P(Z < 2.85)$$

$$= \underline{0.9978}$$





5. Let $W = X - Y$. Find $\text{cov}(X, W)$.

$$\text{cov}(X, W) = E(XW) - E(X)E(W) \quad \text{by cov. formula}$$

$$= E(X(X-Y)) - E(X)E(X-Y)$$

$$= E(X^2 - XY) - E(X)(E(X) - E(Y))$$

$$= E(X^2) - E(XY) - 2(2 - (-1)), \quad \begin{cases} X \sim \text{Normal}(2, 4) \\ Y \sim \text{Normal}(-1, 2). \end{cases}$$

$$= E(X^2) - E(X)E(Y) - 6 \quad \text{by indep. of } X \text{ \& } Y.$$

$$= E(X^2) - (2 * (-1)) - 6$$

$$= E(X^2) + 2 - 6 = E(X^2) - 4$$

$$= \left(V(X) + (E(X))^2\right) - 4. \quad \text{since } V(X) = E(X^2) - (E(X))^2$$

$$= (4 + 2^2) - 4 = 8 - 4 = 4.$$

14.3.3 Independent $R\vec{V}$s. (generalize from bivariate case)

Let $X=(X_1,X_2,\cdots,X_{m_X})$ be a random row vector in $\mathbb{R}^{1\times m_X}$

or a random column vector in $\mathbb{R}^{m_X\times 1}$

with JDF and JPDF given by

$F_{X_1,X_2,\ldots X_{m_X}}$ and $f_{X_1,\ldots,X_{m_X}}$ respectively.

Let $Y=(Y_1,\ldots,Y_{m_Y})$ be a random row vector in $\mathbb{R}^{1\times m_Y}$

or a random column vector in $\mathbb{R}^{m_Y\times 1}$

with JDF and JPDF given by

$F_{Y_1,\ldots,Y_{m_Y}}$ and $f_{Y_1,\ldots,Y_{m_Y}}$ respectively.

Two such $R\vec{V}$s are independent if and only if

for any $(x_1,\cdot,x_m)\in\mathbb{R}^{1\times m_X}$ and $(y_1,\cdots,y_m)\in\mathbb{R}^{1\times m_Y}$

$$f_{X_1,\cdots,X_{m_X},Y_1,\ldots,Y_{m_Y}}(x_1,\ldots,x_{m_X},y_1,\cdots,y_{m_Y})=f_{X_1,\ldots X_{m_X}}(x_1,\ldots,x_{m_X})\,f_{Y_1,\ldots,Y_{m_Y}}(y_1,\ldots,y_{m_Y}).$$

Some important Multivariate Random Vectors.

- multinomial (discrete) $R\vec{V}$
- multivariate Normal (continuous) $R\vec{V}$.

Let $e_1=(1,0)$ and $e_2=(0,1)$ be orthonormal basis vectors in $\mathbb{R}^2$.

Recall component-wise vector addition: $(x_1,x_2)+(y_1,y_2)=(x_1+y_1,x_2+y_2)$

Dfn Indicator function: of a set.

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

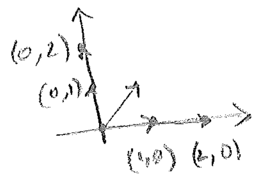
(Show & Tell
Next 2 Lectures)

using slides & animation GUIs etc.
until MLE

Eg $\mathbb{1}_{\{8,9,43\}}(9) = 1$ but $\mathbb{1}_{\{8,9,43\}}(26) = 0$

Ex 14.19

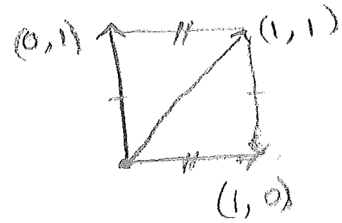
1. $(1,0)+(1,0) = (1+1, 0+0) = (2,0)$



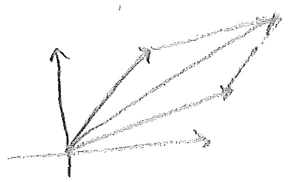
$(1,0)+(0,1) = (1+0, 0+1) = (1,1)$

$(0,1)+(0,1) = (0+0, 1+1) = (0,2)$

2. relationship between $(0,1)$, $(1,0)$ and $(1,1)$ geometrically is.

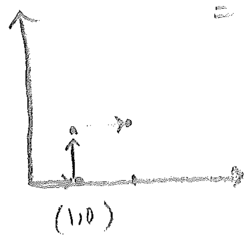


3. diagonal of parallelogram = sum to two sides



4. What is $(1,0)+(0,1)+(1,0)$?

$= (1,1)+(1,0) = (2,1)$

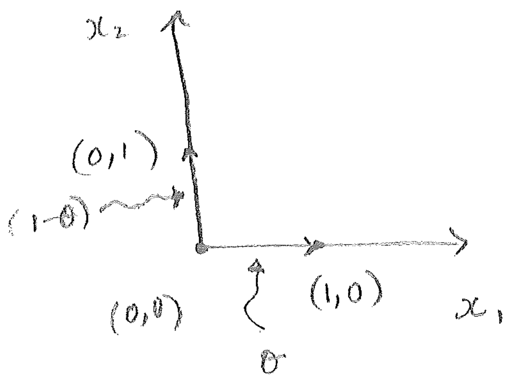


Dfn Bernoulli (θ) $R\vec{V}$

Given a parameter $\theta \in [0,1]$, we say $X := (X_1, X_2)$ is a Bernoulli (θ) $R\vec{V}$ if it has only two possible outcomes in $\{e_1, e_2\} \subseteq \mathbb{R}^2$

JPMF of $X = (X_1, X_2)$ with realisation $x = (x_1, x_2)$ is

$$\underbrace{f_{X_1,X_2}}_{X}(\overbrace{x_1, x_2}^{x}; \theta) = f_X(x;\theta) = \theta \mathbb{1}_{\{e_1\}}(x) + (1-\theta)\mathbb{1}_{\{e_2\}}(x) = \begin{cases} \theta & \text{if } x = e_1 = (1,0) \\ 1-\theta & \text{if } x = e_2 = (0,1) \\ 0 & \text{o.w.} \end{cases}$$



Ex 14.21 What is Expectation of Bernoulli (θ) $R\vec{V}$?

Let $X = (X_1, X_2) \sim$ Bernoulli (θ) $R\vec{V}$.

$$E(X) = E((X_1, X_2)) = \sum_{(x_1,x_2) \in \{e_1, e_2\}} (x_1, x_2) f_{X_1,X_2}(x_1, x_2)$$

$$= (1,0) * \theta + (0,1) * (1-\theta)$$

$$= (\theta, 0) + (0, 1-\theta)$$

$$= (\theta, 1-\theta)$$

Let $Y \sim$ Binomial (n, θ) RV.

We can think of the Binomial (n,θ) $R\vec{V}$ $(Y, n-Y)$

as the sum of n independent Bernoulli (θ) $R\vec{V}s$

$$X_1 = (X_{1,1}, X_{1,2}) \quad , X_2 = (X_{2,1}, X_{2,2}), \cdots \quad , X_n = (X_{n,1}, X_{n,2})$$

that keep track of # heads & # tails in n trials of coin tosses with $P(\text{Heads}) = \theta$.

$$(Y, n-Y) = X_1 + X_2 + \cdots + X_n = (X_{1,1}, X_{1,2}) + (X_{2,1}, X_{2,2}) + \cdots + (X_{n,1}, X_{n,2})$$

Let us generalize the Bernoulli (θ) $R\vec{V}$ to $k \geq 2$ dimensions.

Dfn de Moivre $(\theta_1, \theta_2, \cdots, \theta_k)$ $R\vec{V}$

Given parameters $\theta_1, \ldots, \theta_k$ such that $0 \leq \theta_i \leq 1$ and $\sum_{i=1}^{k} \theta_i = 1$,
and $e_1, e_2, \ldots, e_k$ as orthonormal basis vectors in $\mathbb{R}^k$.

then the JPMF of $R\vec{V}$ $X = (X_1, \cdots, X_k)$ taking a value $x = (x_1, \ldots, x_k)$ is:

$$f_{X_1, \ldots, X_k}((x_1, \ldots, x_k); \theta_1, \ldots, \theta_k) = f_X(x; \theta_1, \ldots, \theta_k)$$

$$= \sum_{i=1}^{k} \theta_i \mathbb{1}_{\{e_i\}}(x) = \begin{cases} \theta_1 & \text{if } x = e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^k \\ \theta_2 & \text{if } x = e_2 = (0, 1, 0, \cdots, 0) \in \mathbb{R}^k \\ \vdots & \\ \theta_k & \text{if } x = e_k = (0, 0, \cdots, 0, 1) \in \mathbb{R}^k \\ 0 & \text{otherwise} \end{cases}$$

When we add n independent de Moivre $(\theta_1, \ldots, \theta_k)$ $R\vec{V}s$,
we get the Multinomial $(n, \theta_1, \ldots, \theta_k)$ $R\vec{V}$.

Dfn: Multinomial $(n, \theta_1, \ldots, \theta_k)$ $\vec{RV}$.

$\vec{RV}$ $Y = (Y_1, \ldots, Y_k)$ obtained from the sum of

n independent de Moivre $(\theta_1, \ldots, \theta_k)$ $\vec{RV}$s with values

$$y = (y_1, \ldots, y_k) \in \left\{ (y_1, \ldots, y_k) \in \mathbb{Z}_+^k : \sum_{i=1}^{n} y_i = n \right\}$$

$\hookrightarrow \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

The set of non-negative integer vectors of length k who elements sum to n.

has JPMF:

$$f(\underbrace{y_1, y_2, \ldots, y_k}_{y}; \underbrace{n, \theta_1, \theta_2, \ldots, \theta_k}_{\text{parameters}}) = \binom{n}{y_1, y_2, \ldots, y_k} \prod_{i=1}^{k} \theta_i^{y_i}$$

where, the multinomial coefficient

$$\binom{n}{y_1, y_2, \ldots, y_k} = \frac{n!}{y_1! \cdot y_2! \cdots y_k!}$$

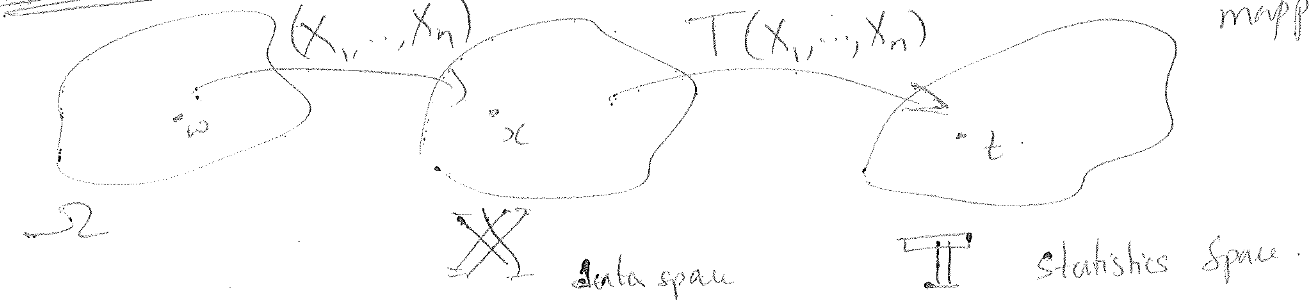
— x —

Let us visualize $\gg$ guiMultinomial

— x —

Also familiarize with Normal (μ, Σ) $\vec{RV}$, the multivariate version of Normal (μ, σ^2) RV.

<u>Statistics</u> are merely functions of data. (usually many-to-one mappings).



Some common statistics are:

1. <u>Sample Mean</u>: realisation

is a RV! $\rightsquigarrow$ $$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \overline{x}_n$$

$$E(\overline{X}_n) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) \underset{\text{if } X_i\text{'s are identically distributed, ie. } E(X_1)=E(X_2)=\cdots=E(X_n)}{=\!=\!=} \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n} n E(X_1) = E(X_1)$$

~~2. Sample Variance~~

$$V(\overline{X}_n) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} V\left(\sum_{i=1}^{n} X_i\right) \underset{\text{if } X_i\text{'s are independently distributed}}{=\!=\!=} \frac{1}{n^2}\sum_{i=1}^{n} V(X_i) = \frac{1}{n^2} n V(X_1) = \frac{1}{n}V(X_1)$$

if X_i's are independently and identically distributed (iid).

also called SRS (Simple Random Sequence) $X_1, \dots, X_n \overset{iid}{\sim} X_1$

2. <u>Sample Variance</u>: $$S_n^2 = \frac{1}{n}\sum_{i=1}^{n} (X_i - \overline{X}_n)^2 \qquad \text{realisation: } s_n^2$$

Once again if $X_1, \dots, X_n \overset{iid}{\sim} X_1$ then $E(S_n^2) = V(X_1)$ $\boxed{\text{Show this is true.}}$

3. <u>Sample standard deviation</u> = $\sqrt{\text{Sample Variance}}$

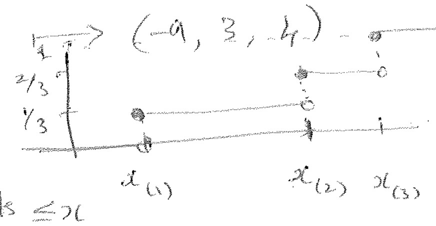
$$S_n = \sqrt{S_n^2} \qquad \text{realisation is } s_n.$$

4. <u>Order statistics</u> (just sort data) $(x_1, x_2, x_3) \mapsto (x_{(1)}, x_{(2)}, x_{(3)})$

$X_{([n])} = (X_{(1)}, X_{(2)}, \dots, X_{(n)})$ where $X_{(1)} \leq X_{(2)} \dots \leq X_{(n)}$.

eg: $(4, 3, -9) \mapsto (-9, 3, 4)$.

5. <u>Empirical Distribution function</u>.

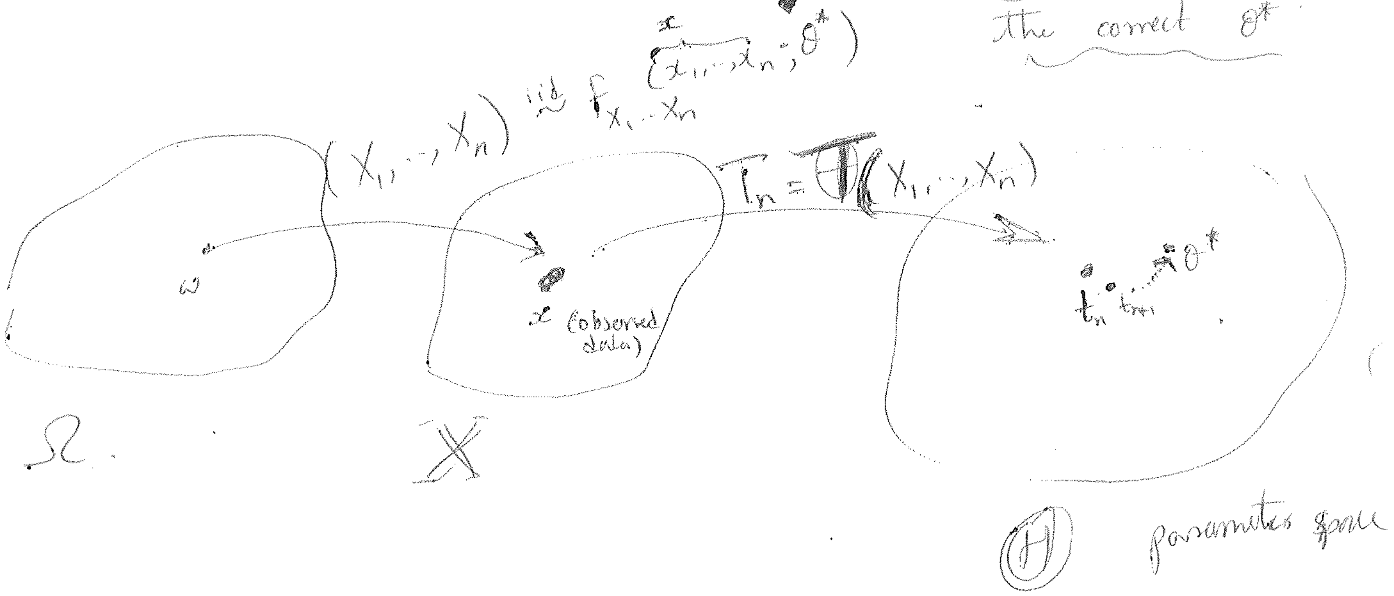


$$\widehat{F}_n(x) = \frac{\#\text{ data points} \leq x}{n}$$

6. Histograms (you know this!)

We can use statistics for Estimating the unknown parameter (in an asymptotically consistent way)

as $n \to \infty$ we want to estimate the correct θ^*



So need notions for convergence or limits of ~~RV~~ sequence of RVs. eg:

$T_1, T_2, \dots \dots$?