§13 Characteristic Functions (CF)

CF of a RV is another way to specify its distribution (instead of PDF/PMF or DF).

Dfn Let X be R.V. and $i = \sqrt{-1}$.

The Characteristic function $\phi_X(t) : \mathbb{R} \to \mathbb{C}$ is given by:

(36) $$\phi_X(t) = E(\exp(itX)) = \begin{cases} \sum_x \exp(itx) f_X(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} \exp(itx) f_X(x)\,dx & \text{if } X \text{ is contin.} \end{cases}$$

Notes:

- encodes the distrn. of X as a complex-valued func of the reals"
- CF with t replaced by $-t$ is the Fourier Transform of f_X for contin. RV. X.

What is CF good for?

① gives a (sometimes) easier way to obtain the k-th moment

$$E(X^k) = \begin{cases} \sum_x x^k f_X(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x^k f_X(x) & \text{if } X \text{ is contin.} \end{cases}$$

② allows analysis of RVs & their transformations

① Obtaining Moments from CFs.

Let X be RV, $\phi_X(t)$ be its CF.

Thm 13.3/4 (Moment & CF) If $E(X^k)$ exists & it's finite, then $\phi_X(t)$ is differentiable k times and (some more cond..)

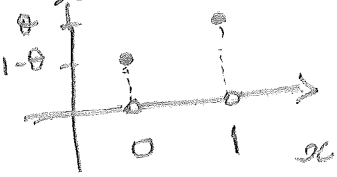
(37) $$\boxed{E(X^k) = \frac{1}{i^k}\left[\frac{d^k \phi_X(t)}{dt^k}\right]_{t=0}}$$

Ex 13.5 (discrete X)

Let $X \sim \text{Bernoulli}(\theta)$. Find CF of X. Then use CF to find $E(X)$, $E(X^2)$ and $V(X)$

Soln. (CF of X). Part I

Recall P.M.F. (X is discrete RV). param $\theta \in (0,1)$.



$$f_X(x;\theta) = \begin{cases} \theta & \text{if } x = 1 \\ 1-\theta & \text{if } x = 0 \\ 0 & \text{o.w.} \end{cases}$$

$$\phi_X(t) = E(\exp(\iota t X)) = \sum_x \exp(\iota t x) f_X(x;\theta)$$

$$= \exp(\iota t * 0)(1-\theta) + \exp(\iota t * 1)\theta$$

$$= \exp(0)(1-\theta) + \exp(\iota t)\theta$$

$$\phi_X(t) = 1 - \theta + \theta e^{\iota t}$$

Part II ($E(X)$ & $E(X^2)$).

Let's differentiate CF:

$$\frac{d}{dt}\phi_X(t) = \frac{d}{dt}\left(1 - \theta + \theta e^{\iota t}\right)$$

$$= 0 - 0 + \theta \iota e^{\iota t} = \underline{\underline{\theta \iota e^{\iota t}}}$$

By (37): $$E(X) = \frac{1}{\iota}\left[\frac{d}{dt}\phi_X(t)\right]_{t=0}$$

$$= \frac{1}{\iota}\left[\theta \iota e^{\iota t}\right]_{t=0} = \frac{1}{\iota}\theta \iota e^{0} = \underline{\underline{\theta}}$$

Aside

Let's check $E(X)$ by direct computation: ✓

$$E(X) = \sum_x x f_X(x;\theta) = [0*(1-\theta)] + [1*\theta] = \theta$$

Again from (27):

$$E(X^2) = \frac{1}{\iota^2}\Big[\underbrace{\frac{d^2}{dt^2}\phi_X(t)}_{\underbrace{\frac{d}{dt}\theta\iota e^{\iota t}}_{\theta\iota^2 e^{\iota t}}}\Big]_{t=0}$$

so. $E(X^2) = \frac{1}{\iota^2}\left[\theta\iota^2 e^{\iota t}\right]_{t=0} = \theta e^0 = \underline{\theta}$

Aside.
Let's check $E(X^2)$ by a direct computation:

$$E(X^2) = \sum_x x^2 f_X(x) = \left[0^2 * (1-\theta)\right] + 1^2 * \theta = \underline{\theta} \checkmark$$

Finally, $V(X) = E(X^2) - (E(X))^2 = \theta - \theta^2 = \theta(1-\theta)$.

Ex 13.6 (contin. X)

Let $X \sim \text{Exponential}(\lambda)$. Find its CF and using it find $E(X)$, $E(X^2)$, $V(X)$.

Recall PDF of X is $f_X(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{o.w} \end{cases}$

Part I (CF).

$$\phi_X(t) = E(e^{\iota X t}) = \int_{-\infty}^{\infty} e^{\iota t x} f_X(x;\lambda)\,dx$$

$$= \cancel{\int_{-\infty}^{0} e^{\iota t x}\, 0\, dx}^{\,0} + \int_0^{\infty} e^{\iota t x}\lambda e^{-\lambda x}\,dx$$

$$= \lambda\int_0^{\infty} e^{-(\lambda-\iota t)x}\,dx$$

$$= \frac{\lambda}{(\lambda-\iota t)}\underbrace{\left|\int_0^{\infty}(\lambda-\iota t)\,e^{-(\lambda-\iota t)x}\,dx\right|}_{=1}$$

$$\phi_X(t) = \frac{\lambda}{\lambda - it}$$

Part 2 (Moments from CF)

Differentiate CF w.r.t. t: $\frac{d}{dt}\phi_X(t) = \frac{d}{dt}\left(\frac{\lambda}{\lambda - it}\right)$

$$= \lambda \frac{d}{dt}\left((\lambda - it)^{-1}\right) = \lambda\left(-(\lambda - it)^{-2}\frac{d}{dt}(\lambda - it)\right)$$

$$= \lambda\left(\frac{-1}{(\lambda - it)^2} * (-i)\right) = \frac{\lambda i}{(\lambda - it)^2}$$

Now (37) gives:

$$E(X) = \frac{1}{i}\left[\frac{d}{dt}\phi_X(t)\right]_{t=0} = \frac{1}{i}\left[\frac{\lambda i}{(\lambda - it)^2}\right]_{t=0}$$

$$= \frac{1}{i}\,\frac{\lambda i}{\lambda^2} = \frac{1}{\lambda}$$

$E(X) = \frac{1}{\lambda}$ ✓ (comp. with Ex 10.9 direct computation).

From (37):

$$E(X^2) = \frac{1}{i^2}\left[\frac{d^2}{dt^2}\phi_X(t)\right]_{t=0}$$

Recall $i^2 = -1$

$$\frac{d}{dt}\left(\frac{\lambda i}{(\lambda - it)^2}\right) = \lambda i \frac{d}{dt}(\lambda - it)^{-2} = \lambda i\left(-2(\lambda - it)^{-3}\underbrace{\frac{d}{dt}(\lambda - it)}_{-i}\right)$$

$$= \frac{2\lambda i^2}{(\lambda - it)^3}$$

$$E(X^2) = \frac{1}{i^2}\left[\frac{2\lambda i^2}{(\lambda - it)^3}\right]_{t=0} = \frac{1}{i^2}\,\frac{2\lambda i^2}{\lambda^3} = \frac{2}{\lambda^2}$$

(Ex 10.9 ✓)

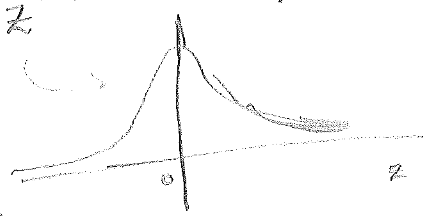
Finally, $V(X) = E(X^2) - (E(X))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$ ✓

(5)

(2) CFs allow analysis of RVs:

Let $Z \sim \text{Normal}(0,1)$

PDF $f_Z(z) = e^{-z^2/2} / \sqrt{2\pi}$



CF of Z:

$$\phi_Z(t) = E\left(e^{itZ}\right)$$

— x — skip? — x —

$$= \int_{-\infty}^{\infty} e^{itz} f_Z(z)\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itz} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itz - z^2/2}\, dz$$

"Trick" 1: (1)

$(z-it)^2 = z^2 - 2zit + (it)^2$

$(z-it)^2 = z^2 - 2zit - t^2$

add t^2 on both sides and divide by -2

$-\left(\frac{(z-it)^2 + t^2}{2}\right) = zit - \frac{z^2}{2}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(t^2 + (z-it)^2)/2}\, dz$$

$$= \frac{e^{-t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(z-it)^2/2}\, dz.$$

"Trick" 2: Subst.

$y = z - it$, $dy = dz$

$$= \frac{e^{-t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy$$

"Trick" 3: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy = 1$

$$= \frac{e^{-t^2/2}}{\cancel{\sqrt{2\pi}}} \cancel{\sqrt{2\pi}}$$

x — skip x — x

$$\boxed{\phi_Z(t) = e^{-t^2/2}} \quad (38).$$

CF of a linear combination

Let $Y = a + bX$, a & b are real constants with $b \neq 0$.

Then CF of Y interms of CF of X is:

$$\boxed{\phi_Y(t) = \exp(\iota a t) \cdot \phi_X(bt)} \qquad (39)$$

—x— skip?

"why."

$$\phi_Y(t) = E(\exp(\iota t Y)) = E(\exp(\iota t (a + bX)))$$

$$= E(\exp(\iota t a + \iota t b X))$$

$$= E(\exp(\iota t a) \exp(\iota t b X))$$

$$= \exp(\iota t a) \, E(\exp(\iota t b X))$$

$$= \exp(\iota t a) \; \phi_X(bt)$$

—x— skip?

More generally, If $X_1, X_2, \ldots, X_n$ are independent RVs. (don't affect each other's outcomes) and

$a_1, a_2, \ldots, a_n$ are some constants,

then CF of $Y = \sum_{i=1}^{n} a_i X_i$ is:

$$\boxed{\phi_Y(t) = \prod_{i=1}^{n} \phi_{X_i}(a_i t) = \phi_{X_1}(a_1 t)\, \phi_{X_2}(a_2 t) \cdots \phi_{X_n}(a_n t)}$$

(39') ④

Ex 13.7

$Y \sim$ Normal (μ, σ^2). Find CF of Y.

Recall (Ex 9.8) $Y = \mu + \sigma Z$, $Z \sim$ Normal $(0,1)$

By (39) $\phi_Y(t) = \exp(\imath\mu t)\, \phi_Z(\sigma t)$

$$= e^{\imath\mu t}\, e^{-(\sigma t)^2/2}$$

$$= e^{\imath\mu t - (\sigma^2 t^2)/2}$$

$Y = a + b X$
$Y = \mu + \sigma Z$

since $\phi_Z(t) = e^{-t^2/2}$

Ex (13.7)'

Let $Y \sim$ Binomial (n, θ). Find CF of Y by using facts: ① $Y = \sum_{i=1}^{n} X_i$, $X_i \sim$ Bernoulli (θ) indep.

② $\phi_{X_i}(t) = (1 - \theta + \theta e^{\imath t})$, from Ex 13.5

③ (39')

Taking $a_1 = a_2 = \cdots = a_n = 1$ in (39'):

$$\phi_Y(t) = \prod_{i=1}^{n} \phi_{X_i}(a_i t) = \prod_{i=1}^{n} 1 - \theta + \theta e^{\imath t} = (1 - \theta + \theta e^{\imath t})^n$$

Ex 13.7''

i.e. $Z + Z \overset{?}{=} 2Z$

Let Z_1 and Z_2 be indep. Normal$(0,1)$ RVs

1. Find CF of $Z_1 + Z_2$. Let $Y = Z_1 + Z_2$

By (39') $\phi_Y = \phi_{Z_1+Z_2} = \phi_{Z_1}(1 \cdot t) + \phi_{Z_2}(1 \cdot t) = e^{-t^2/2}\, e^{-t^2/2} = e^{-t^2}$

2. What RV has CF e^{-t^2}?

ans: $Y \sim$ Normal $(0, 2)$, $\phi_Y = e^{\imath * 0 * t - (2t^2)/2}$

$= e^{-t^2}$ (Ex 13.7)

So $Z_1 + Z_2 \sim$ Normal $(0, 2)$

3. CF of $2Z_1$ is:

By (39'), $\phi_{2Z_1} = \phi_{Z_1}(2t) = e^{-2^2t^2/2} = e^{-2t^2}$

4. The RV $2Z_1$ with CF $e^{-2^2t^2/2} = e^{-2t^2}$

Normal $(0, 4)$ R.V. because its by Ex 13.7

its CF $= e^{i*0*t - (4t^2)/2}$

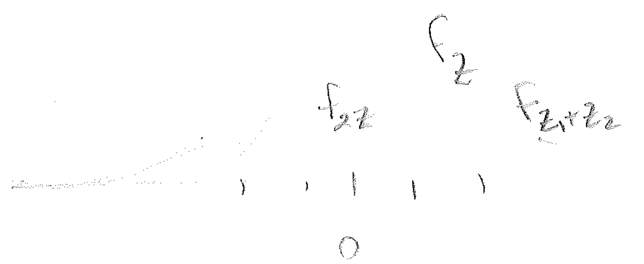
$= e^{-2^2t^2/2} = e^{-2t^2}$.

5. Moral

$2Z_1$ has variance 4

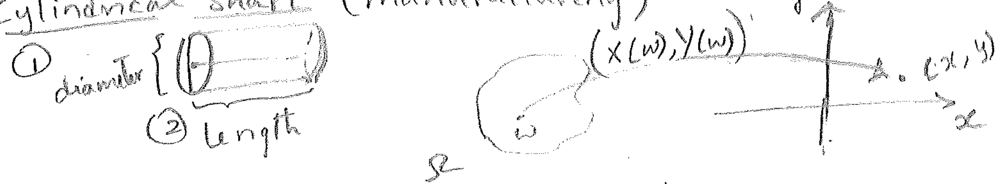
but $Z_1 + Z_2$ has variance 2.

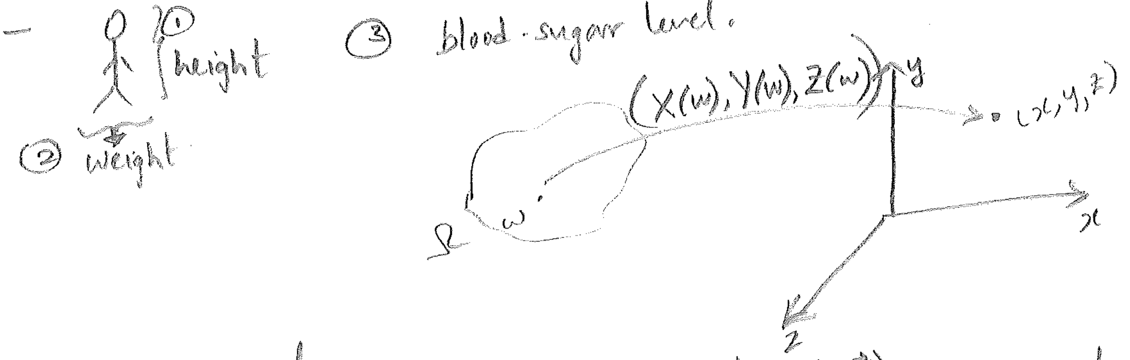
So, $2Z \overset{\text{in distm.}}{\neq} Z_1 + Z$



§14 Random Vectors ($R\vec{V}$)

- Often we measure two or more aspects simultaneously.
  - Cylindrical shaft (manufacturing)





- In general a random vector ($R\vec{V}$) is an m-tuple of random variables.

$$(X_1, \ldots, X_m) : \Omega \longrightarrow \mathbb{R}^m$$

so every $\omega \in \Omega$ is measured by above mapping to be an element of $\mathbb{R}^m$, i.e., $\Omega \ni \omega \mapsto \underbrace{(X_1(\omega), \ldots, X_m(\omega))}_{\in \mathbb{R}^m}$

§14-1 Bivariate Random Vectors.

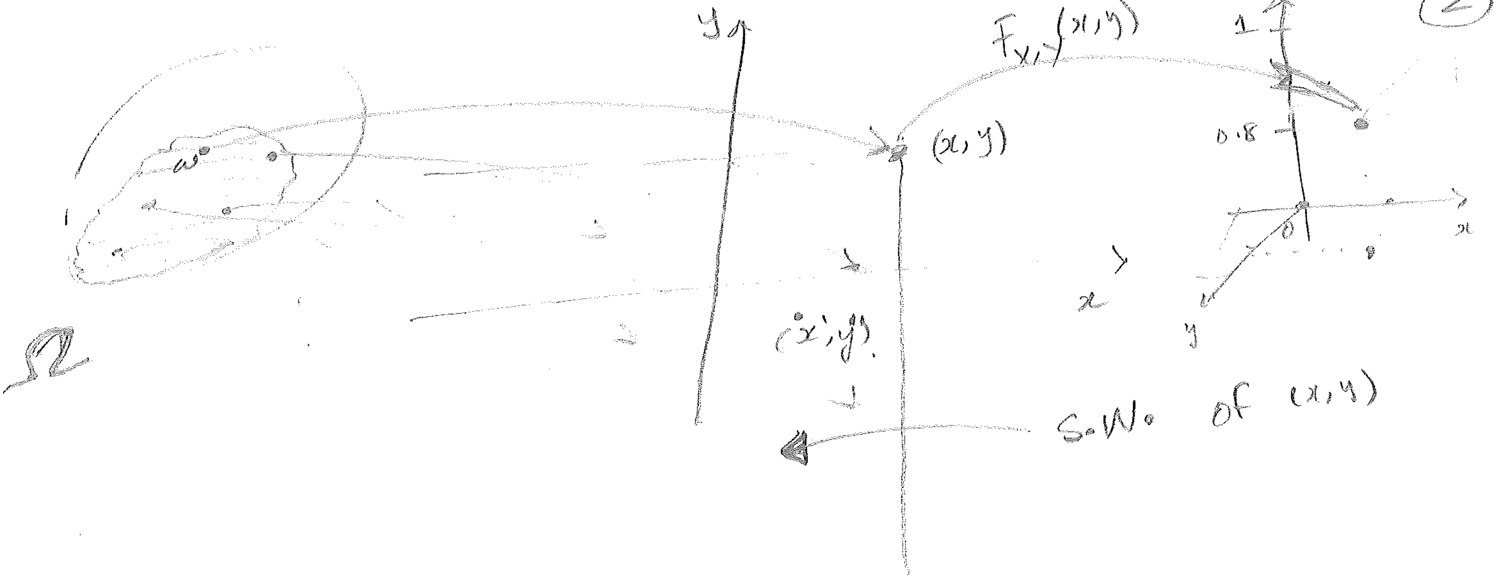
Dfn. The joint distribution function (JDF) or joint cumulative distn. function (JCDF)

$$F_{X,Y}(x, y) : \mathbb{R}^2 \to [0, 1]$$

of the bivariate $R\vec{V}$ (X, Y) is:

(40) $$F_{X,Y}(x, y) = P(X \le x \cap Y \le y) = P(X \le x, Y \le y)$$

$$= P(\{\omega : X(\omega) \le x, Y(\omega) \le y\}), \text{ for any } (x, y) \in \mathbb{R}^2$$



<u>JDF</u> $F_X(x,y): \mathbb{R}^2 \to \mathbb{R}$ <u>satisfies</u> The following <u>properties</u>:

1. $0 \leq F_{X,Y}(x,y) \leq 1$.

2. $F_{X,Y}(x,y)$ is a non-decreasing function of both x & y.

3. $F_{X,Y}(x,y) \to 1$ as $x \to \infty$ and $y \to \infty$

4. $F_{X,Y}(x,y) \to 0$ as $x \to -\infty$ and $y \to -\infty$.

<u>Dfn</u> If (X,Y) is a <u>discrete $\vec{RV}$</u> that takes values in a discrete <u>support</u> <u>set</u>

$$\mathcal{S}_{X,Y} = \{(x_i, y_j) : i = 1,2,\dots \text{ and } j = 1,2,\dots\} \subseteq \mathbb{R}^2$$

with probabilities

$$p_{i,j} = P(X = x_i, Y = y_j) > 0,$$ then the <u>joint</u> <u>Probability</u> <u>mass function</u> (JPMF) of (X,Y) is:

$$f_{X,Y}(x_i, y_j) = P(X = x_i, Y = y_j) = \begin{cases} p_{i,j} & \text{if } (x_i, y_j) \in \mathcal{S}_{X,Y} \\ 0 & \text{otherwise} \end{cases} \qquad (41)$$

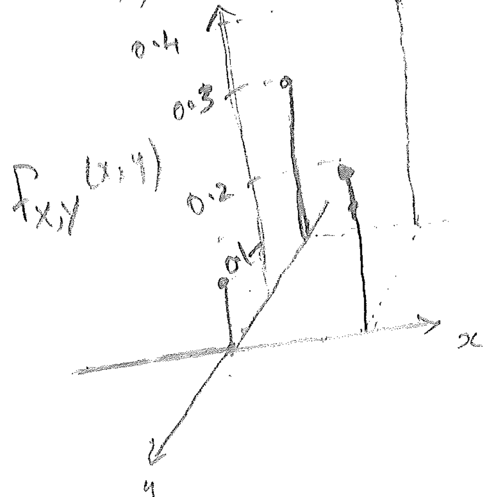
Note that since $P(\Omega)=1$, $\sum_{(x_i,y_j)\in S_{X,Y}} f_{X,Y}(x_i,y_j)=1$

We can get JDF and prob. of any event from JPDF by simply taking sums:

$$F_{X,Y}(x,y)=\sum_{x_i\le x,\, y_j\le y} f_{X,Y}(x_i,y_j), \quad P(B)=\sum_{(x_i,y_j)\in B\cap S_{X,Y}} f_{X,Y}(x_i,y_j) \tag{12}$$

<u>EX. 14.3</u>: Let (X,Y) be a <u>discrete bivariate</u> <u>$\vec{RV}$</u> with the following joint probability mass function (JPMF):

$$f_{X,Y}(x,y)=P(X=x,Y=y)=\begin{cases}0.1 & \text{if } (x,y)=(0,0)\\ 0.3 & \text{if } (x,y)=(0,1)\\ 0.2 & \text{if } (x,y)=(1,0)\\ 0.4 & \text{if } (x,y)=(1,1)\\ 0 & \text{otherwise}\end{cases}$$



Helpful to write JPMF $f_{X,Y}(x,y)$ in tabular form:

	$Y=0$	$Y=1$
$X=0$	0.1	0.3
$X=1$	0.2	0.4

Table reads:
$f_{X,Y}(0,0)=0.1$

1. Let $B=\{(0,0),(1,1)\}$.
$P(B)=?$

(4)

Ans
$$P(B) = \sum_{(x,y)\in \underbrace{\underbrace{\{(0,0),\ (1,1)\}}_{=B} \cap S_{X,Y}}_{=B}} \overbrace{P((X,Y)=(x,y))}^{=P(X=x,\,Y=y)=f_{X,Y}(x,y)} = \sum_{(x,y)\in\{(0,0),(1,1)\}} f_{X,Y}(x,y)$$

$$= f_{X,Y}(0,0) + f_{X,Y}(1,1) = 0.1 + 0.4$$

(2) What is $F_{X,Y}(1/2, 1/2)$?

$$F_{X,Y}(1/2, 1/2) = \sum_{\underbrace{\{(x,y):\, x\le \frac{1}{2},\, y\le\frac{1}{2}\}\cap S_{X,Y}}_{=\{(0,0)\}}} f_{X,Y}(x,y) = f_{X,Y}(0,0) = 0.1$$

(3)
$$F_{X,Y}(3/2, 1/2) = \sum_{\underbrace{\{(x,y):\, x\le 3/2,\, y\le\frac{1}{2}\}\cap S_{X,Y}}_{=\{(0,0),(1,0)\}}} f_{X,Y}(x,y) = F_{X,Y}(0,0) + f_{X,Y}(1,0)$$
$$= 0.1 + 0.2 = 0.3$$

(4)
$$F_{X,Y}(4,5) = \sum_{\underbrace{\{(x,y):\, x\le 4,\, y\le 5\}\cap S_{X,Y}}_{=S_{X,Y}=\{(0,0),(0,1),(1,0),(1,1)\}}} f_{X,Y}(x,y) = f_{X,Y}(0,0) + f_{X,Y}(0,1) + f_{X,Y}(1,0) + f_{X,Y}(1,1)$$
$$= 0.1 + 0.3 + 0.2 + 0.4$$
$$= 1$$

(5)
$$F_{X,Y}(-4,-1) = \sum_{\underbrace{\{(x,y):\, x\le -4,\, y\le -1\}\cap S_{X,Y}}_{\{\}}} f_{X,Y}(x,y) = P(\{\}) = 0.$$

Dfn (X,Y) is a continuous bivariate $R\vec{V}$

if its Joint Distribution function (JDF) $F_{X,Y}(x,y)$ is differentiable and the joint prob density function (JPDF) is given by:

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

From JPDF we can compute JDF at any point $(x,y) \in \mathbb{R}^2$:

$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v)\, du\, dv. \qquad (43)$$

From JPDF we can compute prob. of any event B that can be cast as an integrable region of $\mathbb{R}^2$.

$$P(B) = \iint_B f_{X,Y}(x,y)\, dx\, dy. \qquad (44)$$

JPDF satisfies:

1. integrates to 1. i.e. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(u,v)\, du\, dv.$

2. is a non-negative function. i.e., $f_{X,Y}(x,y) \geq 0$ at every $(x,y) \in \mathbb{R}^2$

Ex 14.5 (project notes).

Let (X,Y) be a continuous $R\vec{V}$ that is uniformly distributed on the unit square $[0,1]^2 = [0,1]\times[0,1]$ with following JPDF :

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,1]^2 \\ 0 & \text{o.w.} \end{cases}$$

Find the following:

(1) JDF $F_{X,Y}(x,y)$ for any $(x,y) \in [0,1]^2$

soln :

Let $(x,y) \in [0,1]^2$, by Eq.(43):

$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\, du\, dv.$$

$$= \int_0^y \int_0^x 1 \, du\, dv = \int_0^y [u]_{u=0}^{x} dv = \int_0^y x \, dv$$

$$= x[v]_{v=0}^{y} = x[y-0] = xy.$$

(2) $P(X \le 1/3,\ Y \le 1/2)$

From (43) & (44) we know $P(X \in (-\infty, 1/3],\ Y \in (-\infty, 1/2])$

$$= P(X \le 1/3,\ Y \le 1/2)$$

$$= \int_{-\infty}^{1/2}\int_{-\infty}^{1/3} f_{X,Y}(u,v)\, du\, dv$$

$$= F_{X,Y}(1/3, 1/2)$$

by part (1)

$$= \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}.$$

alternatively, integrate $f_{X,Y}$ over as per Eq.(44). This amounts to finding the area since "height" = 1

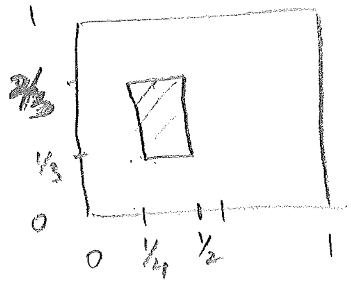
$P(X \le 1/3, Y \le 1/2)$
$= P([0,1/3]\times[0,1/2]) = 1/2 \times 1/3 = 1/6$

Ex. 14.5

3. $P((X,Y) \in [1/4, 1/2] \times [1/3, 2/3])$

soln: Integrate JPDF over the rectangular area

$$\left[\frac{1}{4}, \frac{1}{2}\right] \times \left[\frac{1}{3}, \frac{2}{3}\right]$$



$$= \int_{1/3}^{2/3} \int_{1/4}^{1/2} 1 \, dx \, dy$$

$$= \int_{1/3}^{2/3} \left[x\right]_{1/4}^{1/2} dy = \int_{1/3}^{2/3} \left(\frac{1}{2} - \frac{1}{4}\right) dy = \frac{1}{4}\left[y\right]_{1/3}^{2/3} = \frac{1}{4} \cdot \left[\frac{2}{3} - \frac{1}{3}\right] = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$$

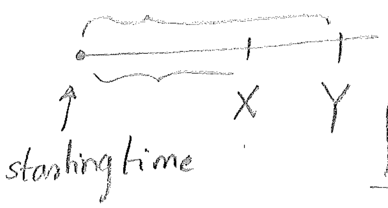
Remark:

For a bivariate uniform $R\vec{V}$ on the unit square $[0,1]^2 = [0,1] \times [0,1]$, the $P(\underbrace{[a,b] \times [c,d]}) = (b-a)(d-c)$ for any event given by the rectangle inside $[0,1]^2$.

Thus prob of an event is equal to its area.

Ex 14.7. Let X = time until a web server connects to your computer (in milliseconds)

Y = time until the server authorizes you as a valid user (in milliseconds)



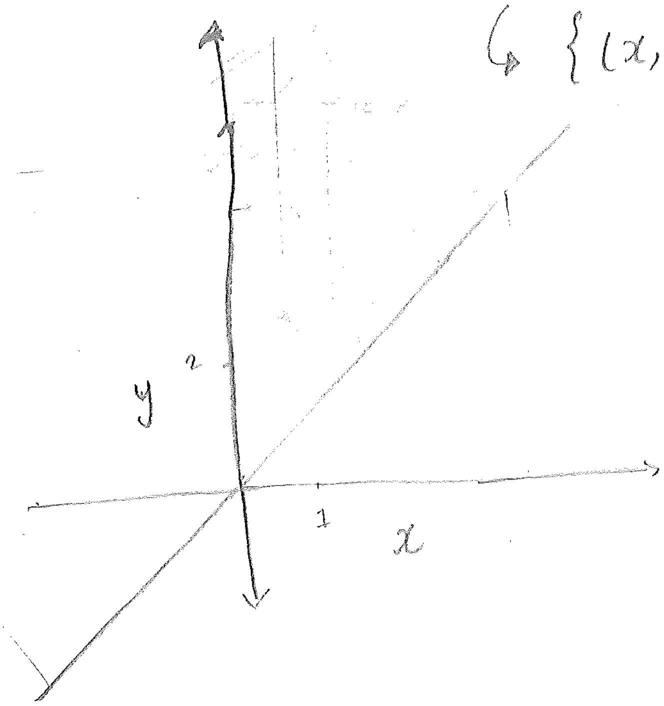
From historical behaviour we know JPDF of $R\vec{V}$ (X,Y) is:

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{10^6} \exp\left(-\frac{1}{1000}x - \frac{2}{1000}y\right), & \text{if } x>0, y>0, x<y \\ 0 & \text{o.w.} \end{cases}$$

So, $X < Y$ (authorization after connection)

Answer The following:

1. Identify the support of (X, Y)



$\hookrightarrow \{(x,y) : f_{X,Y}(x,y) > 0\}$, The region of plane where JPDF is positive

The support is the intersection of

$\left.\begin{array}{l} x > 0 \\ y > 0 \end{array}\right\}$ positive or I quadrant

and $y > x \}$ half plane (above $y = x$ line).

2. Check that $f_{X,Y}$ indeed integrates to 1.

$$\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_{x=0}^{\infty}\int_{y=x}^{\infty} f_{X,Y}(x,y)\,dy\,dx$$

Think!

$$= \frac{6}{10^6}\int_{x=0}^{\infty}\int_{y=x}^{\infty} \exp\left(-\frac{1}{1000}x - \frac{2}{1000}y\right) dy\,dx =$$

$$= \frac{6}{10^6}\int_{x=0}^{\infty}\left(\underbrace{\int_{y=x}^{\infty}\exp\left(-\frac{2}{1000}y\right)dy}\right)\exp\left(-\frac{1}{1000}x\right)dx$$

$$\left[-\frac{1000}{2}\exp\left(-\frac{2}{1000}y\right)\right]_{y=x}^{\infty} = 0 - \left(-\frac{1000}{2}\exp\left(-\frac{2x}{1000}\right)\right) = \frac{1000}{2}\exp\left(-\frac{2}{1000}x\right)$$

2. (contd...)

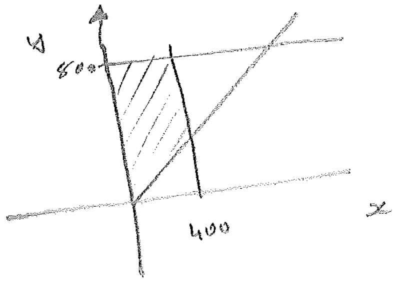
$$= \frac{6}{10^6}\int_{x=0}^{\infty}\left(\frac{1000}{2}\exp\left(-\frac{2}{1000}x\right)\right)\exp\left(-\frac{1}{1000}x\right)dx$$

$$= \frac{6}{10^6}\cdot\frac{10^3}{2}\int_{x=0}^{\infty}\exp\left(-\frac{3}{1000}x\right)dx$$

$$= \frac{3}{10^3}\left[-\frac{1000}{3}\exp\left(-\frac{3}{1000}x\right)\right]_{x=0}^{\infty} = \frac{3}{10^3}\left[-0-\left(-\frac{1000}{3}\underbrace{\exp\left(-\frac{3}{1000}*0\right)}_{=1}\right)\right]$$

$$= \frac{3}{10^3}\left[\frac{1000}{3}\right] = 1.$$

3. Find $P(X \le 400,\ Y \le 800)$



$$P(X \le 400,\ Y \le 800) = \int_{x=0}^{400}\int_{y=x}^{800} f_{X,Y}(x,y)\,dy\,dx.$$

From 2.

$$= \frac{6}{10^6}\int_{x=0}^{400}\left(\underbrace{\int_{y=x}^{800}\exp\left(-\frac{2}{1000}y\right)dy}_{\left[-\frac{1000}{2}\exp\left(-\frac{2}{1000}y\right)\right]_{y=x}^{800}}\right)\exp\left(-\frac{1}{1000}x\right)dx.$$

$$\left[-\frac{1000}{2}\exp\left(-\frac{2}{1000}y\right)\right]_{y=x}^{800} = -\frac{1000}{2}\left[\underbrace{\exp\left(-\frac{1600}{1000}\right)}_{e^{-8/5}} - \exp\left(\frac{-2x}{1000}\right)\right]$$

$$= \frac{6}{10^6}\cdot\frac{10^3}{2}\int_{x=0}^{400}\left(\exp\left(\frac{-2x}{1000}\right) - e^{-8/5}\right)\exp\left(-\frac{1}{1000}x\right)dx$$

$$= \frac{3}{10^3}\int_{x=0}^{400}\exp\left(-\frac{3}{1000}x\right) - e^{-8/5}\exp\left(-\frac{1}{1000}x\right)\,dx$$

$$= \frac{3}{1000}\left(\left[-\frac{1000}{3}\exp\left(-\frac{3}{1000}x\right)\right]_{x=0}^{400} - e^{-8/5}\left[-1000\exp\left(-\frac{1}{1000}x\right)\right]_{x=0}^{400}\right)$$

$$= 3\left(\frac{1}{3}\left[-e^{-6/5} - (-e^{0})\right] - e^{-8/5}\left[-e^{-2/5} - (-e^{0})\right]\right)$$

$$= 3\left(\frac{1}{3}\left(1 - e^{-6/5}\right) - e^{-8/5}\left(1 - e^{-2/5}\right)\right)$$

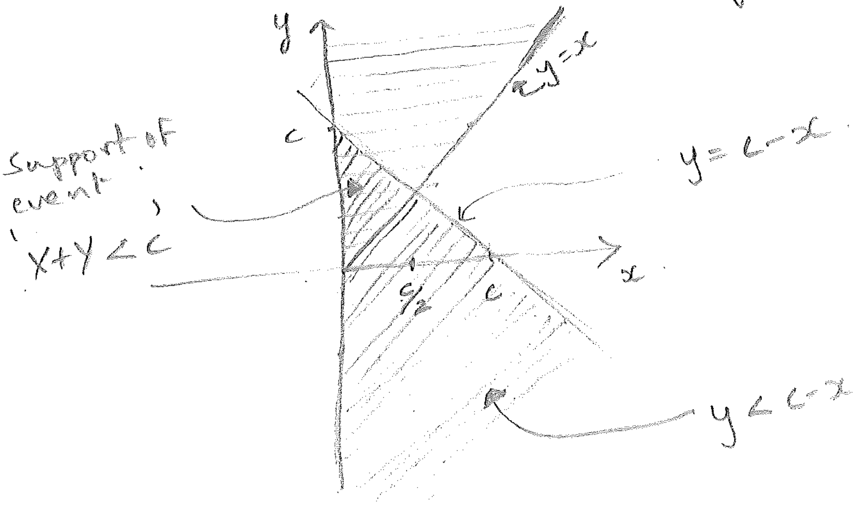
$$\cong 0.499$$

4. Humans prefer response time under $\frac{1}{10}$ secs (10^2 millisecs) from server before they become impatient.

What is $P(X+Y < \underset{\underset{c}{\parallel}}{10^2})$?

- Let's do this more generally for any c.
- First identify the region in the plane where $X+Y<c$

$\boxed{Y < c - X}$



and intersect it with the suppor of $f_{X,Y}$

$x>0,\ y>0,\ y>x$

$$P(X+Y \le c) = \int_{x=0}^{c/2}\int_{y=x}^{c-x} \frac{6}{10^6}\exp\left(-\frac{1}{1000}x - \frac{2}{1000}y\right)dy\,dx$$

$$= \cdots = 1 - 4e^{-3c/2000} + 3e^{-c/500}$$

So, $P(X+Y \le 100) = 1 - 4e^{-300/2000} + 3e^{-100/500} \approx 0.134$ (1 in 10 requests take <

Dfn

If $R\vec{V}$ (X,Y) has $f_{X,Y}(x,y)$ as its JPDF or JPMF then the marginal PDF or PMF of X in $R\vec{V}$ (X,Y) is:

$$f_X(x) = \begin{cases} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy & \text{if } (X,Y) \text{ is continuous } R\vec{V} \\ \sum_y f_{X,Y}(x,y) & \text{if } (X,Y) \text{ is discrete } R\vec{V} \end{cases}$$

and the marginal PDF or PMF of Y in $R\vec{V}$ (X,Y) is:

$$f_Y(y) = \begin{cases} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx & \text{if } (X,Y) \text{ is continuous } R\vec{V} \\ \sum_x f_{X,Y}(x,y) & \text{if } (X,Y) \text{ is discrete } R\vec{V} \end{cases}$$

Ex 14.9 Obtain marginal PMFs $f_Y(y)$ and $f_X(x)$ from JPMF $f_{X,Y}(x,y)$ of $R\vec{V}$ in Ex 14.3.

Soln: Just sum $f_{X,Y}(x,y)$ over x's & y's.

$f_{X,Y}(x,y)$ in tabular form:

	$Y=0$	$Y=1$	
$X=0$	0.1	0.3	$0.4 = f_X(0)$
$X=1$	0.2	0.4	$0.6 = f_X(1)$
	0.3 $= f_Y(0)$	0.7 $= f_Y(1)$	

So,

$$f_X(0) = \sum_y f_{X,Y}(0,y)$$
$$= f_{X,Y}(0,0) + f_{X,Y}(0,1)$$
$$= 0.1 + 0.3$$
$$= 0.4$$

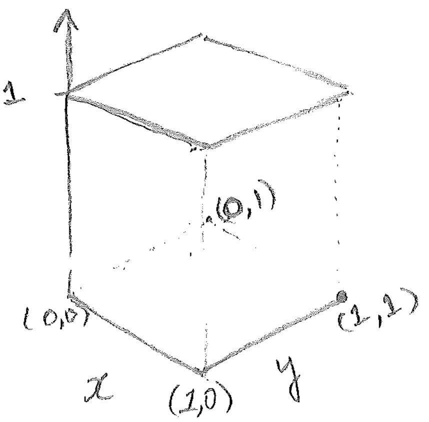
etc.

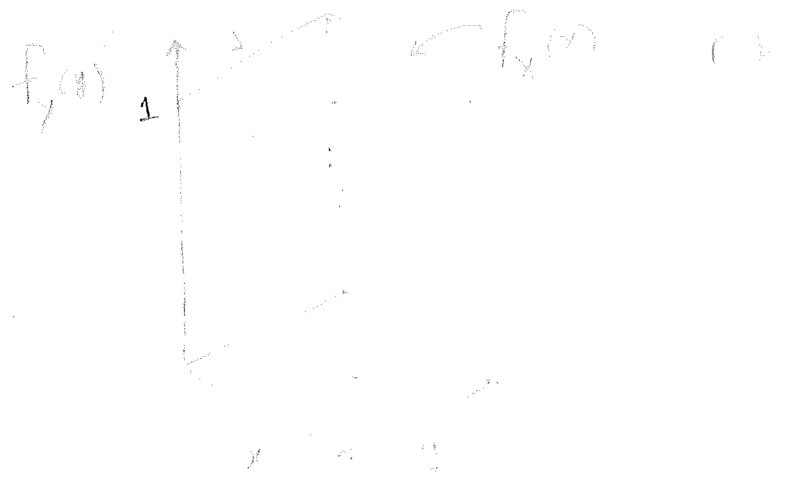
Remark:

Thus, marginal PMF gives the prob. of a specific RV, within a $R\vec{V}$, taking a value irrespective of the value taken by the other RV in this $R\vec{V}$.

Ex 14.10

Obtain the marginal PDFs $f_Y(y)$ and $f_X(x)$ from the JPDF $f_{X,Y}(x,y)$ of continuous $R\vec{V}$ in Ex 14.5 (the bivariate uniform $R\vec{V}$ on $[0,1]^2$).





$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 f_{X,Y}(x,y)\,dy = \int_0^1 1\,dy = \Big[y\Big]_0^1 = 1-0 = 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 1\,dx = \Big[x\Big]_0^1 = 1-0 = 1$$

Ex 14.11 Obtain marginal PDF $f_Y(y)$ from JPDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{10^6}\exp\left(-\frac{1}{1000}x - \frac{2}{1000}y\right) & : \text{if } x>0,\ y>0,\ x<y \\ 0 & : \text{otherwise.} \end{cases}$$

Use $f_Y(y)$ to compute the prob. that Y exceeds 2000 milliseconds.