EMTH 119 Review

Set theory

Experiments

Probability

Cond. Prob

RVs (discrete & continuous)

Transformations of RVs

Expectations of functions of RVs

---

§1 Set theory

A set is a collection of distinct elements.

egs.

$A = \{1, 2, 3\}$ $B = \{1, 8, 3, 6\}$

Note:
$A = \{2, 3, 1\}$
order doesn't matter.

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (integers)

$\mathbb{N} = \{1, 2, 3, \ldots\}$ (natural numbers)

belongs to

$2 \in A$ $8 \notin A$

2 belongs to A 8 doesnot belong to A

2 is an element of A 8 is not an element of A

subset

$\{2, 3\} \subseteq A$ but $\{2, 9\} \not\subseteq A$

not a subset of

$\{2,3\}$ is a subset of A
(since every element in $\{2,3\}$ is an element of A)

equal

$A = B$ if A and B have the same elements

$\{1,2\} = \{2,1\}$ but $\{3,2,1\} \neq \{3,2,8,9\}$

union

$A \cup B$ is the set of elements in A or in B or in both A and B.

$\{1,2\} \cup \{3,2\} = \{1,2,3\}$.

intersection

$A \cap B$ is the set of elements in both A and B.

$\{1,2\} \cap \{3,2\} = \{2\}$.

empty set

$\emptyset = \{\}$, the set with no elements

Universe & complement

Let Ω uppercase omega be a universal set (defining "everything")

and let $A \subseteq \Omega$. Then "A complement" or A^c is the set of all elements in Ω that are not in A.

eg $\Omega = \{H,T\}$, $A = \{H\}$ then $A^c = \{T\}$

Do Ex. 1.1

§2 Experiments.

An experiment is an activity that results in distinct possibilities called outcomes.

Sample space Ω is the set of all outcomes

Subsets of Ω are called events.

A single outcome $\omega \in \Omega$ is called a simple event

Events $E_1, E_2, \ldots, E_n$ that cannot happen simultaneously are called mutually exclusive or pair-wise disjoint events.

eg: Toss a coin experiment

$\Omega = \{\omega_1, \omega_2\}$

or $\Omega = \{H, T\}$

eg: Toss a die experiment $\Omega = \{1, 2, 3, 4, 5, 6\}$.

event $A = \{2, 4, 6\}$ is 'toss an even number'

Trial is a single performance of an experiment

n-product experiment is obtained by repeatedly performing an experiment n times.

eg: toss a coin twice.

sample space $\Omega = \{HH, HT, TH, TT\}$.

event that exactly one 'H' occurs is $\{HT, TH\}$.

§ 3 Background material: Counting the possibilities

$A = \{1, 2, 3, 7\}$ is a *finite* set since it has

$\#A = 4$ elements and $4 < \infty$, infinity.

But $\mathbb{N} = \{1, 2, 3, \ldots\}$ is *countably infinite*.

and $\mathbb{Z} = \{\ldots, -2, -1, 0, +1, +2, +3, \ldots\}$ is also *countably infinite*

However, the set of real numbers $\mathbb{R} = (-\infty, \infty)$

is *uncountably infinite*.

multiplication Principle (product of possibilities)

eg:- # of binary strings of length 5. $2 \times 2 \times$

Factorial: $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$. eg. $3! = 3 \times 2 \times 1 = 6$.

Permutation is an *ordered* selection of n objects.

eg:-

a b c
a c b
b a c
b c a
c a b
c b a

thus there are 6 distinct permutations of a, b, c

Number of distinct permutations of n objects taking r at a time is $\dfrac{n!}{(n-r)!}$

Do. Ex 3.3.

Number of distinct combinations (unordered) of n objects taking r at a time is $\binom{n}{r} = \dfrac{n!}{(n-r)!\, r!}$

Do. Ex. 3.3, 3.4.

Recall Walking in Manhattan

$\binom{2}{2} = 1 = \#\{NN\}$

$\binom{2}{1} = 2 = \#\{EN, NE\}$

$\binom{2}{0} = 1 = \#\{EE\}$

§4 Probability

fn Prob. P is a function which maps event to real numbers in $[0,1]$

$$P : \text{Events} \longrightarrow [0,1]$$

such that.

Axiom 1: for any event A, $0 \leq P(A) \leq 1$.

Ax 2: if Ω is the sample space then $P(\Omega)=1$

Ax 3: if $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$

Ax 4: if $A_1, A_2, \ldots$ is an infinite sequence of mutually exclusive events. (i.e. $A_i \cap A_j = \emptyset$ for any $i \neq j$).

then
$$P\Big(\underbrace{\bigcup_{i=1}^{\infty} A_i}_{A_1 \cup A_2 \cup A_3 \cup \cdots}\Big) = \sum_{i=1}^{\infty} P(A_i)$$

__Meaning__
$$P(A) = \lim_{n \to \infty} \frac{\#\text{ of times } A \text{ occurs in } n \text{ trials}}{n}$$

limiting frequency interpretation of prob.

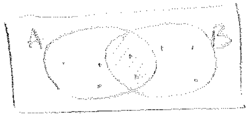
__Rules__ __Complementation Rule__: $P(A^c) = 1 - P(A)$.

__Addition Rule for mutually exclusive events__: $A_1, \ldots, A_n$

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) = P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$
$$= \sum_{i=1}^{n} P(A_i)$$

__Addition Rule for two arbitrary events__: A & B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



__eg__:- Model First Lotto Ball as a well-stirred Urn.
$P(\{37, 17, 27, 7\}) = 4/40$

§5 Conditional Probability

Prob that event B occurs given that event A occurs = $P(B|A) \doteq \dfrac{P(A \cap B)}{P(A)}$, if $P(A) \neq 0$

Conditional Prob is Prob.

Cond. Prob. Satisfies all 4 axioms: (Ax1) For any event B, $0 \le P(B|A) \le 1$

(Ax2) $P(\Omega|A) = 1$

(Ax3) If $B_1 \cap B_2 = \phi$ then
$$P(B_1 \cup B_2|A) = P(B_1|A) + P(B_2|A)$$

(Ax4) For mutually exclusive events $B_1, B_2, \ldots$
$$P(B_1 \cup B_2 \cup \cdots | A) = P(B_1|A) + P(B_2|A) + \cdots$$

Rules

complementation rule $P(B|A) = 1 - P(B^c|A)$.

addition rule. $P(B_1 \cup B_2|A) = P(B_1|A) + P(B_2|A) - P(B_1 \cap B_2|A)$.

multiplication rule If A, B are events and $P(A) \neq 0$, $P(B) \neq 0$, then $P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$.

Do. Ex 5.3

Independent Events. Two events A & B are said to be independent if $P(A \cap B) = P(A)\,P(B)$

[this means $P(A|B) = P(A)$ and $P(B|A) = P(B)$]

Similarly n events are independent if
$$P(A_1 \cap A_2 \cdots \cap A_n) = P(A_1)\,P(A_2) \cdots P(A_n).$$

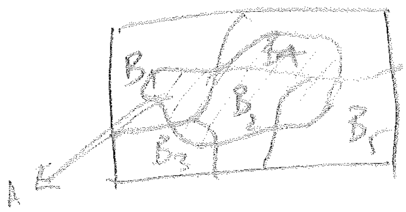
eg:- Toss a fair coin three times (independently): $P(HTH) = P(H)P(T)P(H) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

Total Prob. Thm.

B_i's partition The sample space Ω — Suppose $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$, $P(B_i) > 0$ for each i and $B_i \cap B_j = \emptyset$ for $i \neq j$, then

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

for some arbitrary event A



Ex 5.7

Bayes' Thm

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

$\boxed{\text{Ex } 5.9}$ A = 'a woman has B.C', B = 'test is positive'

$$P(A|B) \xrightarrow{\text{Bayes Thm}} = \frac{P(A)P(B|A)}{P(B)} = \frac{0.01 \times 0.9}{\underbrace{P(A^c \cap B)}_{0.099} + \underbrace{P(A \cap B)}_{0.009}} = \frac{9}{108}$$

Total Prob. Thm.

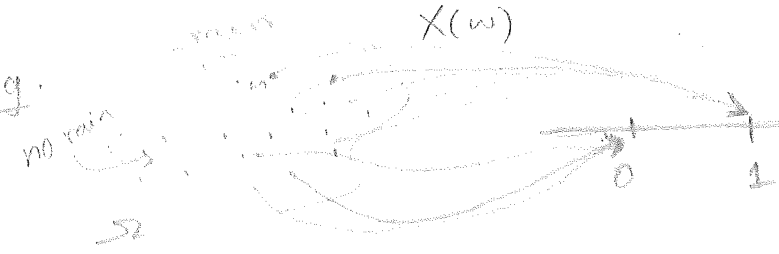
Prob. that a woman has B.C. given that she tested positive

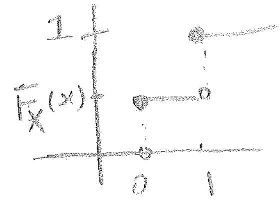
§6 Random Variables

A random variable X is a function from the sample space Ω to real line

$$X : \Omega \to \mathbb{R}$$

eg.

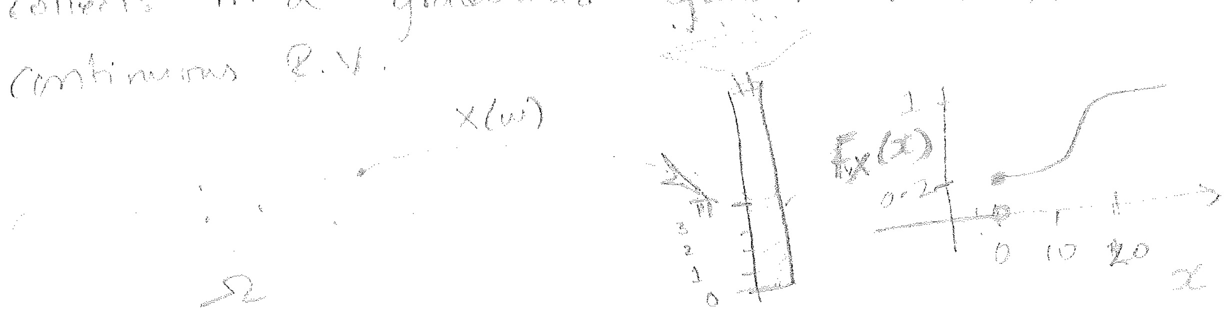




$\Omega = \{\text{rain}, \text{no rain}\}$. $X(\omega) = \begin{cases} 1 & \text{if } \omega = \text{rain} \\ 0 & \text{if } \omega = \text{no rain} \end{cases}$

This is a discrete R.V.

eg: If we measure the amount of rain that collects in a graduated cylinder then X is a continuous R.V.



<u>Distribution</u> <u>Function</u> $F_X : \mathbb{R} \to [0,1]$ of R.V. X

is

$$F_X(x) = P(X \leq x) = P(\{w : X(w) \leq x\}) \quad \text{for any } x \in \mathbb{R}$$

<u>Meaning</u>

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

<u>Rules</u>

(i) $0 \leq F_X(x) \leq 1$.

(ii) F_X is a non-decreasing function of x

(iii) $F_X(x) \to 1$ as $x \to +\infty$

$F_X(x) \to 0$ as $x \to -\infty$

<u>Continuous R.V. X</u>

If $F_X(x)$ is differentiable then $f_X(x) = F_X'(x)$ is called Probability density function (PDF).

Hence

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

<u>Properties of PDF</u>

(i) $f_X(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

(iii) $P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$.

Population Moments (continuous R.V. X)

Dfns

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

completing the square

$$V(X) \stackrel{\text{def}}{=} E(X - E(X))^2 = E(X^2) - (E(X))^2$$

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

Rules: $E(aX+b) = a\,E(X) + b$
$V(aX+b) = a^2 V(X)$

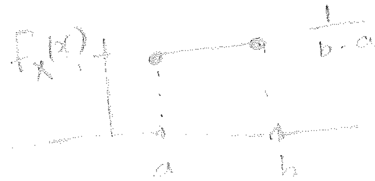
Three common contin. RVs

parameters

① $X \sim \text{Uniform}(a, b)$, $a < b$

PDF

$$f_X(x;a,b) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{ow} \end{cases}$$



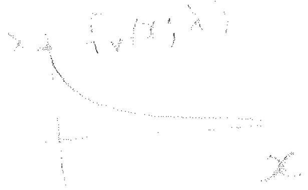
DF

$$F_X(x;a,b) = \begin{cases} 0 & \text{if } -\infty < x < a \\ \frac{x-a}{b-a} & \text{if } a \le x < b \\ 1 & \text{if } b \le x < +\infty \end{cases}$$

② $X \sim \text{Exponential}(\lambda)$ parameter $\lambda > 0$

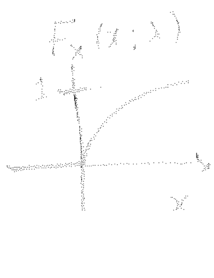
PDF

$$f_X(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$$



DF

$$F_X(x;\lambda) = \int_{-\infty}^{x} f_X(u;\lambda)\,du = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$$

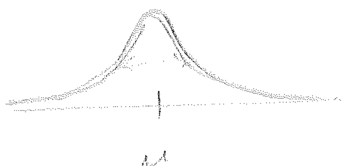


(2)

(3) $X \sim \text{Normal}(\mu, \sigma^2)$,   <u>Parameters</u>: $\mu \in (-\infty, +\infty)$, $\sigma^2 > 0$.

<u>PDF</u>

$$f_X(x;\mu,\sigma^2) = \frac{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{\sqrt{2\pi\sigma^2}}, \quad -\infty < x < \infty$$



$$E(X) = \mu \quad , \quad V(X) = \sigma^2$$

---- * ---- * SKIP * ---- SKIP ---- * ----

<u>Example</u> (warm-up)

Let $X \sim \text{Exponential}(2)$ RV (i.e. param $\lambda = 2$).

Find the following:

(a) $f_X(x;2)$, the PDF of X

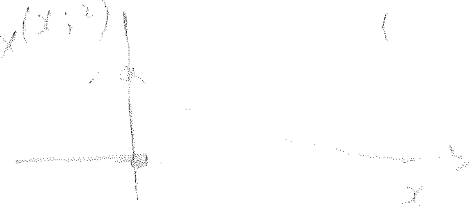
(b) $F_X(x;2)$

(c) $E(X)$

(d) $V(X)$

(e) $P(2 < X < 5)$

<u>Soln.</u> (a) $f_X(x;2) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$

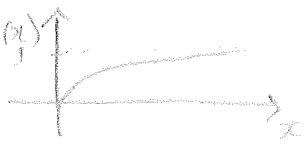


(b) $F_X(x;2) = \int_{-\infty}^{x} f_X(u)\,du$

if $x \le 0$ then $F_X(x;2) = \int_{-\infty}^{x} 0\,du = 0$

if $x > 0$ Then $F_X(x) = \int_0^x 2e^{-2u}\,du = \left(-e^{-2u}\right]_0^x = 1 - e^{-2x}$

So, $F_X(x;2) = \begin{cases} 1 - e^{-2x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$



(c) $E(X) = \int_0^\infty x\, f_X(x;2)\, dx = \int_0^\infty x\, 2e^{-2x}\, dx$

I.B.P.

$$\int_0^\infty u(x)\, dv(x) = \left(u(x)\, v(x)\right]_0^\infty - \int_0^\infty v(x)\, du(x)$$

$$\boxed{v(x)} = \int dv(x) = \int 2e^{-2x} dx = -e^{-2x}$$

$$du(x) = 1\, dx$$

$$= \left(x\left(-e^{-2x}\right)\right]_0^\infty - \int_0^\infty -e^{-2x}\, 1\, dx$$

$$= 0 + \int_0^\infty e^{-2x} dx = \left(-\frac{e^{-2x}}{2}\right]_0^\infty = \frac{1}{2}$$

(d) $E(X^2) = \int_0^\infty x^2 f_X(x;2)\, dx = \int_0^\infty x^2\, 2e^{-2x} dx$

I.B.P.

$$= \left(x^2\left(-e^{-2x}\right)\right]_0^\infty - \int_0^\infty 2x\left(-e^{-2x}\right) dx$$

$$= 0 + 2\int_0^\infty x e^{-2x} dx = \frac{1}{2}$$

(c)

then $V(X) = E(X^2) - (E(X))^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

(e) $P(2 < X < 5) = F_X(5;2) - F_X(2;2)$

$$= (1 - e^{-10}) - (1 - e^{-4}) = e^{-4} - e^{-10}.$$

—X— **END SKIP** —X—

Read!! Ex 10.9 p. 62

If $X \sim$ Exponential (λ) RV. with any parameter $\lambda > 0$,

then $E(X) = \frac{1}{\lambda}$, $E(X^2) = \frac{2}{\lambda^2}$, $V(X) = \frac{1}{\lambda^2}$

(14)

§12 Approximate Expectations of Functions of RVs

Suppose $Y = g(X)$.

we Want approximations of

$$\underbrace{E}_{\text{Expectation}}\big(\underbrace{g}_{\text{function of } X}(\underbrace{X}_{\text{R.V.}})\big) = \int_{-\infty}^{\infty} g(x)\underbrace{f_X(x)}_{\text{PDF of } X}\,dx$$

We will get it by Delta Method

based on Taylor series approximation to $Y = g(X)$

Recall Taylor series for $g(x)$ about a is:

$$g(X) = \sum_{i=0}^{n} \underbrace{g^{(i)}(a)}_{i^{th} \text{ derivative of } g \text{ evaluated at } a} \frac{(X-a)^i}{i!} + \text{remainder}$$

(and $g^{0}(a) = g(a)$)

a polynomial in x with coefficients $\frac{g^{(i)}(a)}{i!}$

is used to approximate $g(X)$ locally around a

[animation on sage ?]

Delta Method [Approximating Expectation: $E(g(X))$]

Take Taylor series approximation about $a = E(X) = \mu$

(I) First consider only first two terms

$$g(X) = g(\mu) + g^{(1)}(\mu)(X-\mu) + \text{remainder eq}(2)$$

This gives approximation. $g(X) \underset{\text{approx. equal to}}{\approx} g(\mu) + g'(\mu)(X-\mu)$ Eq(3')

Now, by taking $E(\cdot)$ on both sides of (3')

$\frac{5}{①}$

$$E(g(X)) \approx E\left(g(\mu) + g^{(1)}(\mu)(X-\mu)\right)$$

$$= E(g(\mu)) + E\left(g^{(1)}(\mu)(X-\mu)\right)$$

$$= g(\mu) + g^{(1)}(\mu)\, E(X-\mu)$$

$$= g(\mu) + g^{(1)}(\mu)\, \underbrace{\left[E(X) - E(\mu)\right]}_{\mu \,-\, \mu \,=\, 0}$$

$$= g(\mu)$$

Thus, we get "one-term approximation"

$$\boxed{E(g(X)) \approx g(\mu)} \quad (33)$$

—x—

A better approximation by considering another Taylor term

$$g(X) \approx g(\mu) + g^{(1)}(\mu)(X-\mu) + g^{(2)}(\mu)\frac{(X-\mu)^2}{2}$$

Taking $E(\cdot)$ on both sides:

$$E(g(X)) \approx \underbrace{E(g(\mu)) + E\left(g^{(1)}(\mu)(X-\mu)\right)}_{g(\mu) \text{ from } (33)} + E\left(g^{(2)}(\mu)\frac{(X-\mu)^2}{2}\right)$$

$$= g(\mu) + \frac{1}{2} g^{(2)}(\mu)\, \underbrace{E\left((X-\mu)^2\right)}_{V(X)}$$

Thus, we get "two-term approximation"

$$\boxed{E(g(X)) \approx g(\mu) + \frac{V(X)}{2} g^{(2)}(\mu)} \quad (34)$$

Delta Method [Approximating Variance: $V(g(X))$]

One-term approximation to $V(g(X))$ by taking $V(\cdot)$ on both sides of (31):

$$V(g(X)) \approx V\left(g(\mu) + g^{(1)}(\mu)(X-\mu)\right)$$
$$= V(g(\mu)) + \left(g^{(1)}(\mu)\right)^2 \left(V(X-\mu)\right)$$
$$= 0 + \left(g^{(1)}(\mu)\right)^2 \left(V(X) - \underbrace{V(\mu)}_{=0}\right)$$

Therefore

$$\boxed{V(g(X)) \approx \left(g^{(1)}(\mu)\right)^2 V(X)} \quad (35)$$

Example 12.1 Let $X \sim \text{Uniform}(0,1)$ R.V.

Find one-term approximation of $E(\sin(X))$

From Ex 10.7 we know: $E(X) = \frac{1-0}{2} = \frac{1}{2} = \mu$

$$V(X) = \frac{(1-0)^2}{12} = \frac{1}{12}$$

From (33) $\boxed{E(g(X)) \approx g(\mu)}$ with $g(x) = \sin(x)$.

Therefore,

$$E(\underbrace{\sin(X)}_{g(X)=\sin(X)}) \approx \underbrace{\sin(E(X))}_{g(\mu)} = \sin\left(\tfrac{1}{2}\right) \approx \underline{0.48}$$

Find two-term approx. of $E(\sin(X))$

$$g^{(2)}(\mu) = \left[\frac{d}{dx}\left(\frac{d}{dx}\sin(x)\right)\right]_{x=\mu} = \left(-\sin(x)\right)_{x=\mu}$$

From (34) $\boxed{E(g(X)) \approx g(\mu) + \frac{V(X)}{2}\, g^{(2)}(\mu)}$

Therefore,

$$E(\sin(X)) \approx \sin(\mu) + \frac{1}{2} * \frac{1}{12}\left(-\sin(\mu)\right)$$
$$= \sin\left(\tfrac{1}{2}\right)\left[1 - \tfrac{1}{24}\right] \approx \underline{0.46}$$

For this simple problem we can find $E(\sin(X))$ exactly (for comparison):

$$E(\sin(X)) = \int_{-\infty}^{\infty} \sin(x)\, \underbrace{f_X(x;0,1)}_{=\begin{cases}1 & \text{over } [0,1] \\ 0 & \text{o.w.}\end{cases}} dx = \int_{-\infty}^{0} \cancel{\sin(x)\cdot 0\, dx} + \int_{0}^{1} \sin(x) 1\, dx + \int_{1}^{+\infty} \cancel{\sin(x)\, 0 . dx}$$

$$= \int_0^1 \sin(x)\, dx = \Big(-\cos(x)\Big]_0^1 = -\cos(1) - (-\cos(0))$$

$$= -\cos(1) + 1$$

$$\approx 0 \cdot 45969769 \ldots$$

So, our approximation is decent and improves with additional terms.

Example 12.2 Let $X \sim$ Exponential (λ) R.V. for a given parameter $\lambda > 0$.

Find an approximation of $V\left(\frac{1}{1+X}\right)$

Need to use (35)

$$V(g(X)) \approx \left(g^{(1)}(\mu)\right)^2 V(X)$$

with $g(X) = \frac{1}{1+X}$.

$$V\left(\frac{1}{1+X}\right) \approx \left(-(1+\mu)^{-2}\right)^2 V(X)$$

$$= \left(-\left(1+\frac{1}{\lambda}\right)^{-2}\right)^2 \frac{1}{\lambda^2}$$

good enough for full marks in exam.

$$= \left(1+\frac{1}{\lambda}\right)^{-4} \frac{1}{\lambda^2}$$

$$= \left(\frac{\lambda+1}{\lambda}\right)^{-4} \frac{1}{\lambda^2}$$

$$= \frac{\lambda^4}{(\lambda+1)^4\, \lambda^2} = \frac{\lambda^2}{(1+\lambda)^4}$$

WORK

1 $g^{(1)}(x) = \frac{d}{dx} g(x)$

$$= \frac{d}{dx}\left(\frac{1}{1+x}\right)$$

$$= \left(-1\,(1+x)^{-2}\right) \times 1$$

$$= -(1+x)^{-2}$$

so. $g^{(1)}(\mu) = \left[g^{(1)}(x)\right]_{x=\mu}$

$$\boxed{g^{(1)}(\mu) = -(1+\mu)^{-2}}$$

From Ex 10.9, we know

2 $\boxed{V(X) = \frac{1}{\lambda^2}}$ $\boxed{E(X) = \mu = \frac{1}{\lambda}}$

Example 12.3

Approximate mean & Variance of $Y = g(X) = 2X - 3$

can use Delta Method blindly.

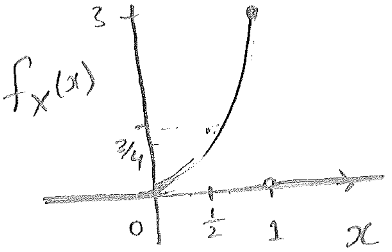
but can get it exactly by realizing g is a linear func. of X.

$$E(Y) = E(2X-3)$$
$$= E(2X) - E(3)$$
$$= 2E(X) - 3$$

$$V(Y) = V(2X-3)$$
$$= V(2X) - V(3)$$
$$= 2^2 \cdot V(X) - 0$$
$$= 4\,V(X)$$

Ex 12.4

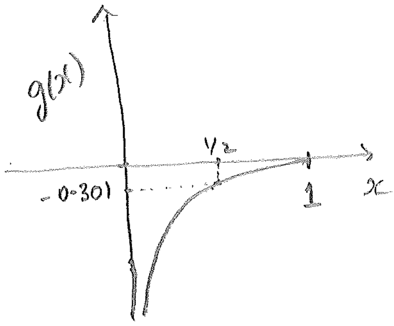
Suppose X is a R.V. with P.D.F.



$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

no parameters so no need for ';'

Find two-Term approx. of $E(Y)$ where, $Y = \underbrace{10 \log_{10}(X)}_{g(X)}$



From (34)

$$\boxed{E(g(X)) \approx g(\mu) + \frac{V(X)}{2}\, g^{(2)}(\mu)}$$

So, we need: $\mu = E(X)$, $V(X)$, $g^{(2)}(\mu)$

$$\mu = E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x \cdot f_X(x)\,dx + \int_{1}^{\infty} 0\,dx$$

$$= \int_0^1 x \cdot 3x^2\,dx = \int_0^1 3x^3\,dx = \left(\frac{3}{4}x^4\right]_0^1$$

$$= \frac{3}{4}\left(1^4 - 0^4\right) = \underline{\underline{\frac{3}{4}}}.$$

To get $V(X)$ we need $E(X^2)$,

$\boxed{\text{Recall: } V(X) = E(X^2) - (E(X))^2}$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 x^2\, 3x^2\,dx$$

$$= \int_0^1 3x^4\,dx = \left(\frac{3}{5}x^5\right]_0^1 = \frac{3}{5}\left(1^5 - 0^5\right) = \underline{\underline{\frac{3}{5}}}$$

$$\text{So, } V(X) = E(X^2) - (E(X))^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16}$$

$$= \frac{3*16 - 9*5}{5*16} = \frac{48-45}{50+30}$$

$$= \underline{\underline{\frac{3}{80}}}$$

Finally we need: $g^{(2)}(\mu) = \left(\frac{d}{dx}\left[\frac{d}{dx} g(x)\right]\right]_{x=\mu}$

$$\frac{d}{dx} g(x) = \frac{d}{dx}\left(10 \log_{10}(x)\right) = \frac{d}{dx}\left(10 \frac{\log_e(x)}{\log_e(10)}\right) = \left(\frac{10}{\log(10)} \frac{1}{x}\right)$$

$$g^{(2)}(x) = \frac{d}{dx}\left(\frac{d}{dx} g(x)\right) = \frac{d}{dx}\left(\frac{10}{\log(10)} \frac{1}{x}\right) = \frac{10}{\log(10)} \frac{d}{dx}\left(x^{-1}\right) = \frac{10}{\log(10)} \cdot \left(-x^{-2}\right)$$

$$g^{(2)}(\mu) = \left(g^{(2)}(x)\right]_{x=\mu=3/4} = \left(-\frac{10}{\log(10)} \frac{1}{x^2}\right]_{x=3/4} = \frac{-10}{\log(10)} \frac{1}{\left(\frac{3}{4}\right)^2}$$

Finally from (34)

$$E(g(X)) \approx g(\mu) + \frac{V(X)}{2} g^{(2)}(\mu)$$

we get

$$E\left(10 \log_{10}(X)\right) \approx 10 \log_{10}\left(\frac{3}{4}\right) + \frac{\left(\frac{3}{80}\right)}{2} \left(\frac{-10}{\log(10)\left(\frac{3}{4}\right)^2}\right) \approx \underline{-1.394}$$

full marks in exam.

By doing the integral directly,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \underbrace{\int_0^1 10 \log_{10}(x)\, 3x^2\, dx}_{\left[\left(\frac{10}{3\log(10)}\right)\left(x^3(3\log(x)-1)\right)\right]_0^1} = \underline{-1.458}$$

— x —

So, we can use such approximation generally when $\sigma = \sqrt{V(X)}$, the std. dev. of X is small compared to $\mu = E(X)$, the pop. mean of X